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FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ or ft	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
-x EE	Deduct x marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	B2,1,0	<p>If not B2 then award B1 for either</p> $[\mathbf{N}=] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } [\mathbf{M}=] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ <p>B2 cannot be awarded if the correct matrix for NM is obtained by using an incorrect matrix for N or M or if found by calculating MN</p>
		2	

	Question 1 Total	2	
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Q	Answer	Marks	Comments
2(a)	$\alpha + \beta + \gamma = 4$ $\alpha\beta + \beta\gamma + \gamma\alpha = 3$	B1	For either $\alpha + \beta + \gamma = 4$ or $\alpha\beta + \beta\gamma + \gamma\alpha = 3$ seen or used
	$\alpha^2 + \beta^2 + \gamma^2$ $= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ $= 4^2 - 2(3)$	M1	Correct formula seen/used
	$\alpha^2 + \beta^2 + \gamma^2 = 16 - 6 = 10$	A1	CSO AG Must be convincingly shown
		3	

Q	Answer	Marks	Comments
2(b)	β is a root of the [cubic] equation	E1	oe
		1	

Q	Answer	Marks	Comments
2(c)	$\alpha^3 + \beta^3 + \gamma^3$ $= 4(\alpha^2 + \beta^2 + \gamma^2) - 3(\alpha + \beta + \gamma) - 3c$	M1	oe in terms of c and expressions whose values have been stated/found eg $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \alpha\gamma) + 3(-c)$
	$= 4(10) - 3(4) - 3c = 40 - 12 - 3c = 28 - 3c$	A1	AG Must be convincingly shown
		2	

	Question 2 Total	6	
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Q	Answer	Marks	Comments
3	$10 = \det(\mathbf{A})\det(\mathbf{B})$ $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1; \quad 5\det(\mathbf{A}) = 1$ $\det(\mathbf{A}) = \frac{1}{5}$ $\det(\mathbf{B}) = 50$ Volume of $S_2 = 6 \times 50 = 300 \text{ [cm}^3\text{]}$	<p>M1</p> <p>A1</p> <p>A1ft</p> <p>B1ft</p>	<p>Either $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ seen/used or $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$ seen/used</p> <p>Correct value for $\det(\mathbf{A})$ seen/used PI by use of $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$</p> <p>ft $10 \div$ their value for $\det(\mathbf{A})$</p> <p>ft $6 \times$ their $\det(\mathbf{B})$</p>
		4	
	Question 3 Total	4	

Q	Answer	Marks	Comments
4(a)	$y = x\sqrt{x} - \frac{1}{3}\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\sqrt{x} - \frac{1}{6\sqrt{x}}$	M1	ACF At least one correct term for $\frac{dy}{dx}$
	$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4}x - \frac{1}{2} + \frac{1}{36x}$ $= \left(\frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}}\right)^2 = \left(\frac{3}{2}x^{\frac{1}{2}} + \frac{1}{6}x^{-\frac{1}{2}}\right)^2$	A1	A correct expression for $1 + \left(\frac{dy}{dx}\right)^2$ in the form $(px^n + qx^{-n})^2$
		2	

Q	Answer	Marks	Comments
4(b)	$S = 2\pi \int_{[1]}^{[4]} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$		
	$= 2\pi \int_{[1]}^{[4]} \left(x\sqrt{x} - \frac{1}{3}\sqrt{x}\right) \left(\frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}}\right) [dx]$	M1	Substitutes into a correct formula for the surface area S ; ft their derivative. Condone missing brackets
	$= 2\pi \int_1^4 \left(\frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18}\right) [dx]$	M1	Integrand $ax^2 + bx + c$ with at least one correct coefficient
	$= 2\pi \left[\frac{x^3}{2} - \frac{x^2}{6} - \frac{x}{18} \right]_1^4$	A1	
	$= 2\pi \left[\left(\frac{64}{2} - \frac{16}{6} - \frac{4}{18}\right) - \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{18}\right) \right]$ $= 2\pi \left(\frac{63}{2} - \frac{15}{6} - \frac{3}{18}\right) = \pi \left(63 - \frac{16}{3}\right) = \frac{173}{3}\pi$	A1	AG Must be convincingly shown
		4	

	Question 4 Total	6	
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Q	Answer	Marks	Comments
5	$\int 9x^2 \ln x \, dx = 3x^3 \ln x - \int 3x^3 \left(\frac{1}{x}\right) \, dx$ $\int 9x^2 \ln x \, dx = 3x^3 \ln x - x^3 \quad [+c]$ $\int_0^e 9x^2 \ln x \, dx = \lim_{a \rightarrow 0} \int_a^e 9x^2 \ln x \, dx$ $= (3e^3 - e^3) - \lim_{a \rightarrow 0} (3a^3 \ln a - a^3)$ $\lim_{a \rightarrow 0} (a^3 \ln a) = 0$ $\int \frac{4}{1+4x^2} \, dx = \int \frac{1}{\frac{1}{4} + x^2} \, dx$ $= 2 \tan^{-1}(2x) \quad [+c]$ $\int_0^e \left(9x^2 \ln x + \frac{4}{1+4x^2} \right) \, dx$ $= 2e^3 + 2 \tan^{-1}(2e)$	<p>M1</p> <p>A1</p> <p>M1</p> <p>B1</p> <p>M1 A1</p> <p>A1</p>	$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 9x^2 \Rightarrow v = 3x^3 \quad \text{PI}$ <p>Correct integration of $9x^2 \ln x$</p> <p>Evidence of limit 0 having been replaced by a (oe) at any stage and $\lim_{a \rightarrow 0}$ seen or taken at any stage with no remaining \lim relating to e</p> <p>Accept if stated in the more general format.</p> <p>M1: $k \tan^{-1}(kx)$ or $\lambda \tan^{-1}(kx)$ A1: $2 \tan^{-1}(2x)$</p> <p>$2e^3 + 2 \tan^{-1}(2e)$</p> <p>Must have scored all previous M and A marks and no errors when substituting limits.</p>
		7	
	Question 5 Total	7	

Q	Answer	Marks	Comments
6	<p>I.F. is $e^{\int(8x(x^2+2)^{-1})dx} = e^{4\ln(x^2+2)}$</p> <p>I.F. $= (x^2+2)^4$</p> $(x^2+2)^4 \frac{dy}{dx} + 8x(x^2+2)^3 y$ $= 2x^3(x^2+2)^4 + \frac{(x^2+2)^4}{(x^2+2)^{\frac{9}{2}}}$ $(x^2+2)^4 y = \int \left(2x^3(x^2+2)^4 + \frac{1}{\sqrt{x^2+2}} \right) dx$ <p>Let $u = x^2 + 2$</p> $\Rightarrow \int 2x^3(x^2+2)^4 dx = \int (u-2)u^4 du$ $\int 2x^3(x^2+2)^4 dx = \frac{(x^2+2)^6}{6} - \frac{2(x^2+2)^5}{5} [+A]$ $\int \left(\frac{1}{\sqrt{x^2+2}} \right) dx = \sinh^{-1}\left(\frac{x}{\sqrt{2}}\right) [+B]$ $y = \frac{(x^2+2)^2}{6} - \frac{2(x^2+2)}{5} + \frac{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right) + c}{(x^2+2)^4}$	<p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>B1</p> <p>A1</p>	<p>I.F. identified and integration attempted</p> <p>Correct integrating factor</p> <p>Multiplying both sides of the given DE by their I.F. and integrating LHS to get $y \times$ I.F.</p> <p>A relevant substitution or relevant integration by parts used to find an expression for the integral of $2x^3(x^2+2)^4$</p> <p>PI by correct integration</p> <p>oe Correct integration of $2x^3(x^2+2)^4$</p> <p>eg $\frac{x^2(x^2+2)^5}{5} - \frac{(x^2+2)^6}{30} [+k]$ or $\frac{1}{6}x^{12} + \frac{8}{5}x^{10} + 6x^8 + \frac{32}{3}x^6 + 8x^4 [+c]$</p> <p>Correct integration of $\frac{1}{\sqrt{x^2+2}}$</p> <p>oe, such as $\ln(x + \sqrt{x^2+2}) [+B]$</p> <p>Correct GS</p> <p>$y = f(x)$ with ACF for $f(x)$</p>
		7	
	Question 6 Total	7	

Q	Answer	Marks	Comments
7(a)	$\sum_{r=1}^n (u_{r+1} - u_r) = \sum_{r=1}^n (2u_r) + \sum_{r=1}^n 4$ LHS = $u_2 - u_1 + u_3 - u_2 + u_4 - u_3 + \dots + u_n - u_{n-1} + u_{n+1} - u_n$ $= u_{n+1} - u_1$ RHS = $\sum_{r=1}^n (2u_r) + 4n$ $2\sum_{r=1}^n (u_r) + 4n = u_{n+1} - u_1; \quad 2\sum_{r=1}^n (u_r) = u_{n+1} - 4n - 3$ $\Rightarrow \sum_{r=1}^n u_r = \frac{1}{2}u_{n+1} - 2n - \frac{3}{2}$	M1 B1 A1	Uses method of differences with $\sum_{r=1}^n (u_{r+1} - u_r) = \sum_{r=1}^n (2u_r) + \sum_{r=1}^n 4$ $\sum_{r=1}^n 4 = 4n \quad \text{or} \quad \sum_{r=1}^n 2 = 2n$ AG Must be convincingly shown
		3	

Q	Answer	Marks	Comments
7(b)	When $n = 1$, $u_1 = 5 \times 3^0 - 2 = 5 \times 1 - 2 = 5 - 2 = 3$ [Formula is true for $n = 1$] Assume formula true for $n = k$ (*), [integer $k \geq 1$,] so $u_{k+1} = 3(5 \times 3^{k-1} - 2) + 4$ $u_{k+1} = 5 \times 3^k - 6 + 4; \quad u_{k+1} = 5 \times 3^{(k+1)-1} - 2$ Hence formula is true for $n = k + 1$ (**) and since true for $n = 1$ (***), formula $u_n = 5 \times 3^{n-1} - 2$ is true for $n = 1, 2, 3, \dots$ by induction (****)	B1 M1 A1 E1	Correct values to show formula true for $n = 1$ Assumes formula true for $n = k$ and considers $u_{k+1} = 3(5 \times 3^{k-1} - 2) + 4$ oe Be convinced Must have (*), (**), (***), present, previous 3 marks scored and a final statement (****) clearly indicating that it relates to positive integers and 'induction'
		4	

Q	Answer	Marks	Comments
7(c)	$\sum_{r=1}^n u_r = \frac{1}{2}(5 \times 3^n - 2) - 2n - \frac{3}{2}$ $= \frac{5}{2} \times (3^n - 1) - 2n$	B1	ACF Accept unsimplified
		1	

	Question 7 Total	8	
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Q	Answer	Marks	Comments
8(a)	Aux. equation $m^2 + 2m = 0$; $m = -2, 0$	M1	Forming and solving the correct aux. equation.
	$y = A + B e^{-2x}$	A1	PI by correct values of m seen/used Correct general solution
		2	

Q	Answer	Marks	Comments
8(b)	$[y_{CF} = A + B e^{-2x}]$; $y_{PI} = a x e^{-2x}$	M1	$y_{PI} = a x e^{-2x}$ seen or used
	$y'_{PI} = a e^{-2x} - 2a x e^{-2x}$	M1	y'_{PI} and y''_{PI} both of the form $\pm c e^{-2x} \pm d x e^{-2x}$
	$y''_{PI} = -4a e^{-2x} + 4a x e^{-2x}$		
	$-4a e^{-2x} + 4a x e^{-2x} + 2a e^{-2x} - 4a x e^{-2x} = 6e^{-2x}$ $\Rightarrow a = -3$	M1	Substitution into the DE to form an equation in x and solve to find a value for a
	$[y_{PI}] = -3 x e^{-2x}$	A1	
	$[y_{GS} =] A + B e^{-2x} - 3 x e^{-2x}$ $A + B = 0$; $4B + 12 = 4$ (or $-2B - 3 = 1$) $A = 2$ and $B = -2$	B1ft A1	Their CF + their PI with exactly two arbitrary constants $A = 2$ and $B = -2$
$y = 2 - 2e^{-2x} - 3 x e^{-2x}$ When $x = 3$ $y = 2 - 11e^{-6}$	A1	$2 - 11e^{-6}$ oe	
		7	

	Question 8 Total	9	
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Q	Answer	Marks	Comments
9(a)	$\det \mathbf{A} = 2k + 12$	B1	Correct expression for $\det \mathbf{A}$
	Cofactor matrix		
	$\begin{bmatrix} 1 & -3 & 4 \\ 3k+12 & -3k & 2k-12 \\ -4k-17 & 4k+3 & 16-2k \end{bmatrix}$	M1	One complete row or column or diagonal correct
	$\begin{bmatrix} 1 & -3 & 4 \\ 3k+12 & -3k & 2k-12 \\ -4k-17 & 4k+3 & 16-2k \end{bmatrix}$	A2	All nine entries correct else A1 for at least six entries correct
	Inverse matrix $\mathbf{A}^{-1} =$	M1	Transpose of their cofactors with no more than one further error and division by their $\det \mathbf{A} (\neq 0)$
	$= \frac{1}{2k+12} \begin{bmatrix} 1 & 3k+12 & -4k-17 \\ -3 & -3k & 4k+3 \\ 4 & 2k-12 & 16-2k \end{bmatrix}$	A1	Correct \mathbf{A}^{-1} scores 6 marks
		6	

Q	Answer	Marks	Comments
9(b)(i)	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	M1	$\mathbf{A}^{-1} \mathbf{v}$ for their \mathbf{A}^{-1} with at least one ft component correct.
	$= \frac{1}{2k+12} \begin{bmatrix} 1+3k+12-4k-17 \\ -3-3k+4k+3 \\ 4+2k-12+16-2k \end{bmatrix}$	A1ft	At least two ft components correct.
	$x = -\left(\frac{k+4}{2k+12}\right) \quad y = \frac{k}{2k+12} \quad z = \frac{8}{2k+12}$	A1	ACF All three correct
		3	

Q	Answer	Marks	Comments
9(b)(ii)	$0 < x + y + z < \frac{1}{3}$	B1	
		1	

	Question 9 Total	10	
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Q	Answer	Marks	Comments
10(a)(i)	$e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}} = 2\cos\left(\frac{\theta}{2}\right)$	B1	
		1	

Q	Answer	Marks	Comments
10(a)(ii)	$\frac{1}{e^{i\theta} + 1} = \frac{e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}} =$ $= \frac{\cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)} = \frac{1}{2} - \frac{i}{2} \tan\left(\frac{\theta}{2}\right)$	M1 A1	<p>Either $\frac{1}{e^{i\theta} + 1} = \frac{e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}}$</p> <p>or $i \tan\left(\frac{\theta}{2}\right) = \left(\frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}} \right)$ seen/used</p> $\frac{1}{2} - \frac{i}{2} \tan\left(\frac{\theta}{2}\right) = \frac{1}{2} - \frac{1}{2} \left[\frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}} \right]$ <p>AG Must be convincingly shown</p>
		2	

Q	Answer	Marks	Comments
10(b)	$\frac{1}{e^{i(\pi-\theta)} + 1} = \frac{1}{2} - \frac{i}{2} \tan\left(\frac{\pi - \theta}{2}\right) = \frac{1}{2} - \frac{i}{2} \cot\left(\frac{\theta}{2}\right)$ $\frac{1}{-e^{-i\theta} + 1} = \frac{1}{2} - \frac{i}{2} \cot\left(\frac{\theta}{2}\right)$ $\frac{1}{-e^{-i\theta} + 1} - 1 = -\frac{1}{2} - \frac{i}{2} \cot\left(\frac{\theta}{2}\right)$ $\frac{e^{-i\theta}}{1 - e^{-i\theta}} = -\frac{1}{2} - \frac{i}{2} \cot\left(\frac{\theta}{2}\right)$ $\frac{1}{e^{i\theta} - 1} = -\frac{1}{2} - \frac{i}{2} \cot\left(\frac{\theta}{2}\right)$	M1 A1	<p>$e^{i\pi} = -1$ seen or used at any stage</p> <p>AG Must be convincingly shown</p>
		3	

Q	Answer	Marks	Comments
11(a)	$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 5 \end{bmatrix}$	B1	$\begin{bmatrix} 10 \\ 10 \\ 5 \end{bmatrix}$ oe
		1	

Q	Answer	Marks	Comments
11(b)	$\left(\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \right) \cdot \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix} =$ $= -40 + 30 - 10 [= -20] \neq 0 \text{ so they are not coplanar vectors}$	M1 A1ft	Correct method to evaluate a relevant scalar triple product; ft their (a) Condone a miscopy of sign from given vectors Must see correct ft evaluation and explicit comparison with 0, or relevant initial statement involving 0, followed by a correct ft conclusion.
		2	

Q	Answer	Marks	Comments
11(c)	Direction vector for L : $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ Direction of normal to the plane: $\mathbf{n} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\mathbf{n} \cdot \mathbf{d} = 4 + 6 - 6$ $\sqrt{2^2 + 2^2 + 1^2} \sqrt{2^2 + 3^2 + (-6)^2} \cos \theta = 4$ $\cos \theta = \frac{4}{21}$ Angle between L and $\Pi = 90^\circ - \cos^{-1} \left(\frac{4}{21} \right)$ $= 11^\circ$ to nearest degree	B1ft M1 m1 A1	Seen or used; \mathbf{d} correct and ft their answer to part (a) for \mathbf{n} in a simplified or unsimplified form Finds a numerical expression for the scalar product of their normal to Π and their \mathbf{d} PI ft their \mathbf{n} and \mathbf{d} PI CAO 11° (Condone missing $^\circ$)
		4	

Q	Answer	Marks	Comments
11(d)	Line $L: \mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ General point on L is $(1+2t, 3t, 2-6t)$ $1+2t = 1-2\lambda + \mu$ $3t = -2 + \lambda - 3\mu; \quad 8t = -4 - 5\mu$ $2-6t = 3+2\lambda + 4\mu; \quad 3-4t = 4+5\mu$ $t = -\frac{3}{4}$ $P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$ $PQ = \sqrt{1.5^2 + 2.25^2 + 4.5^2} = 5.25$	M1 M1 A1 A1	Finding a general point on L Eliminating \mathbf{r} for L and Π , forming and solving three equations in three unknowns (or substituting general pt on L into $2x+2y+z=1$) to find a value for t Correct coordinates or position vector for P PI or $\left -\frac{3}{4}\right \times \sqrt{2^2 + 3^2 + (-6)^2}$ Correct distance for PQ
		4	

Q	Answer	Marks	Comments
11(d) ALT	Plane $\Pi: \mathbf{r} = \begin{bmatrix} 1-2\lambda + \mu \\ -2 + \lambda - 3\mu \\ 3+2\lambda + 4\mu \end{bmatrix}$ Sub in $L: \begin{bmatrix} -2\lambda + \mu \\ -2 + \lambda - 3\mu \\ 1+2\lambda + 4\mu \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \mathbf{0}$ $9-12\lambda + 6\mu = 0$ $2-8\lambda + 14\mu = 0$ $4-8\lambda + 9\mu = 0$ $\mu = \frac{2}{5} \quad \lambda = \frac{19}{20}$ $P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$ $PQ = \sqrt{1.5^2 + 2.25^2 + 4.5^2} = 5.25$	M1 M1 A1 A1	Eliminating \mathbf{r} for L and Π , forming at least two equations in two unknowns Solving three correct equations to obtain a value for each of the unknowns Correct coordinates or position vector for P PI or $\left -\frac{3}{4}\right \times \sqrt{2^2 + 3^2 + (-6)^2}$ Correct distance for PQ
		4	

Q	Answer	Marks	Comments
11(e)	Let T be the point on Π such that QT is perpendicular to Π . Q is a point on line L so angle PQT , is the angle θ from part (c) between normal to Π and line L . $QT = PQ \cos \theta$	M2	oe eg $QT = PQ \sin(90^\circ - \theta)$ condoning their rounded answer for $90^\circ - \theta$ found in part (c)
	$\text{Shortest distance} = 5.25 \times \frac{4}{21} = 1$	A1	CAO Do not accept 1 from non-exact values
		3	

Q	Answer	Marks	Comments
11(e) ALT	$1 + 2t = 1 - 2\lambda + \mu$ $2t = -2 + \lambda - 3\mu; \quad 6t = -4 - 5\mu$ $2 + t = 3 + 2\lambda + 4\mu; \quad 3 + 3t = 4 + 5\mu$ $t = -\frac{1}{3}$ $T\left(\frac{1}{3}, -\frac{2}{3}, \frac{5}{3}\right) \quad Q(1, 0, 2)$ $QT = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$	M1	Finds a general point on QT and eliminates \mathbf{r} for QT and Π , forming and solving three equations in three unknowns oe to find a value for t
	$\text{Shortest distance} = 1$	m1 A1	Finds coordinates of T and uses distance formula to find a value for the distance QT oe CAO Must be from exact values.
		3	

	Question 11 Total	14	
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Q	Answer	Marks	Comments
12(a)	$y = e^{\frac{7}{25}x} (\cosh x)^{-1},$ $\frac{dy}{dx} = \frac{7}{25} e^{\frac{7}{25}x} (\cosh x)^{-1} - e^{\frac{7}{25}x} (\cosh x)^{-2} \sinh x$ <p>At P, $\frac{dy}{dx} = 0 \Rightarrow \frac{7}{25} \cosh x - \sinh x = 0$</p> $\tanh x = \frac{7}{25}$ $x = \tanh^{-1}\left(\frac{7}{25}\right) = \frac{1}{2} \ln\left(\frac{1+\frac{7}{25}}{1-\frac{7}{25}}\right)$ $x = \ln\left(\frac{4}{3}\right)$	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p>	<p>Use of the product formula or quotient formula.</p> <p>ACF</p> <p>Using $\tanh^{-1}(n) = \frac{1}{2} \ln\left(\frac{1+n}{1-n}\right)$</p> <p>$x = \ln\left(\frac{4}{3}\right)$</p>
		6	

Q	Answer	Marks	Comments
12(b)	$y_p = e^{\frac{7}{25}(\ln k)} \operatorname{sech}(\ln k)$ $\operatorname{sech}(\ln k) = \frac{2k}{k^2 + 1}; e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$ $\text{Shortest distance} = \left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}}$	<p>M1</p> <p>B1ft</p> <p>A1</p>	<p>Attempts to find the y-coordinate of P, ft their value of k from part (a)</p> <p>$e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$ and $\operatorname{sech}(\ln k) = \frac{2k}{k^2 + 1}$</p> <p>oe ft their k value</p> <p>oe but must be in the required printed form eg $\left(\frac{18}{25}\right)\left(\frac{4}{3}\right)^{\frac{32}{25}}$</p>
		3	

Q	Answer	Marks	Comments
12(c)	$\text{Shortest distance} \left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}} = 1.04\dots$ <p>Since $[-1 <] \tanh x < 1$, and line L, $y = 1.04\dots$ is above $y = 1$, L does not intersect the curve $y = \tanh x$</p>	<p>B1</p> <p>E1ft</p>	<p>1.04...</p> <p>States $\tanh x < 1$ and if (their shortest distance) < 1 states L intersects $y = \tanh x$ oe if (their shortest distance) ≥ 1 states L does not intersect $y = \tanh x$ oe</p>
		2	

	Question 12 Total	11	
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Q	Answer	Marks	Comments
13(a)	$[\ln(1+4x) =] 4x - 8x^2 + \frac{64}{3}x^3$	B1	Ignore higher order terms
	[valid for] $-\frac{1}{4} < x \leq \frac{1}{4}$	B1	
		2	

Q	Answer	Marks	Comments
13(b)(i)	$y = \ln(\cos x - \sin x);$ $\frac{dy}{dx} = \frac{-\sin x - \cos x}{\cos x - \sin x} = 1 - \frac{2\cos x}{\cos x - \sin x};$	B1	ACF for the first derivative
	$\frac{d^2y}{dx^2} =$ $\frac{2\sin x(\cos x - \sin x) + 2\cos x(-\sin x - \cos x)}{(\cos x - \sin x)^2}$	M1	Quotient rule used
	$\frac{d^2y}{dx^2} = \frac{-2(\sin^2 x + \cos^2 x)}{\cos^2 x + \sin^2 x - 2\sin x \cos x}$ $\frac{d^2y}{dx^2} = \frac{-2}{1 - \sin 2x}$	A1	AG Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(b)(ii)	$\frac{d^3y}{dx^3} = \frac{0 + 2(-2\cos 2x)}{(1 - \sin 2x)^2}$	B1	ACF for the third derivative
	$y(0) = 0; y'(0) = -1; y''(0) = -2; y'''(0) = -4$	M1	All 4 attempted with at least 2 correct
	$\ln(\cos x - \sin x) = 0 - 1x + \frac{(-2)x^2}{2!} + \frac{(-4)x^3}{3!}$ $\ln(\cos x - \sin x) = -x - x^2 - \frac{2}{3}x^3$	A1	AG Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(c)	$\ln\left[(1-\sin 2x)\sqrt{1+4x}\right]=$ $= \ln(1-\sin 2x) + \ln\sqrt{1+4x}$ $= 2\ln(\cos x - \sin x) + \frac{1}{2}\ln(1+4x)$ $= 2\left(-x - x^2 - \frac{2}{3}x^3\right) + \frac{1}{2}\left(4x - 8x^2 + \frac{64}{3}x^3\right)$ $\lim_{x \rightarrow 0} \left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2 + 6x^3} \right]$ $= \lim_{x \rightarrow 0} \left[\frac{-6x^2 + \frac{28}{3}x^3 \dots}{5x^2 + 6x^3} \right]$ $= \lim_{x \rightarrow 0} \left[\frac{-6 + \frac{28}{3}x \dots}{5 + 6x} \right] \quad \text{[so the limit exists]}$ $\lim_{x \rightarrow 0} \left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2 + 6x^3} \right] = -\frac{6}{5}$	<p>M1</p> <p>M1</p> <p>A1ft</p> <p>M1</p> <p>A1</p>	<p>Seen or used</p> <p>$\ln(1-\sin 2x) = 2\ln(\cos x - \sin x)$ seen or used</p> <p>ft their 3-term expansion in part (a)</p> <p>Substitutes series expansions and divides numerator and denominator by x^2 to reach the form</p> <p>$\lim_{x \rightarrow 0} \left[\frac{P + O(x)}{Q + O(x)} \right]$, so limit exists $= \frac{P}{Q}$</p> <p>CSO</p>
		5	
	Question 13 Total	13	

Q	Answer	Marks	Comments
14(a)	$B\left(6+3\sqrt{2}, \frac{7\pi}{4}\right); \quad OA = OB = 6+3\sqrt{2}$ Angle $AOB = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$ Area of triangle $AOB =$ $\frac{1}{2}(6+3\sqrt{2})^2 = \frac{1}{2}(54+36\sqrt{2}) = 27+18\sqrt{2}$	M1 A1	Uses polar coordinates of A and B oe to find two relevant lengths and an angle to use in finding the required area PI AG Must be convincingly shown
		2	

Q	Answer	Marks	Comments
14(b)	$r = \frac{3}{2} \operatorname{cosec}^2\left(\frac{\theta}{2}\right); \quad r = \frac{3}{1-\cos\theta}$ $r - r \cos\theta = 3; \quad r = 3+x$ $r^2 = (3+x)^2; \quad x^2 + y^2 = (3+x)^2$ $y^2 = 6x+9$	M1 M1 M1 A1	$r = \frac{3}{1-\cos\theta}$ condone sign error Use of $r \cos\theta = x$ to eliminate θ $r^2 = x^2 + y^2$ used at any stage $y^2 = 6x+9$ oe for $f(x)$
		4	

Q	Answer	Marks	Comments
14(c)(i)	$\tan \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}, \theta = \frac{4\pi}{3}$ When $\theta = \frac{\pi}{3}, r = 6$ When $\theta = \frac{4\pi}{3}, r = 2$ Polar coordinates of P and Q are $\left(6, \frac{\pi}{3}\right), \left(2, \frac{4\pi}{3}\right)$	B2,1,0 B1	B2 any three correct; else B1 for any two correct $\left(6, \frac{\pi}{3}\right), \left(2, \frac{4\pi}{3}\right)$
		3	

Q	Answer	Marks	Comments
14(c)(ii)	$\int \left(1 + \cot^2\left(\frac{\theta}{2}\right)\right) \operatorname{cosec}^2\left(\frac{\theta}{2}\right) d\theta$ Let $u = \cot\left(\frac{\theta}{2}\right), \frac{du}{d\theta} = -\frac{1}{2} \operatorname{cosec}^2\left(\frac{\theta}{2}\right)$ $\int \operatorname{cosec}^4\left(\frac{\theta}{2}\right) d\theta = \int (1 + u^2)(-2) du$ $= -2 \cot\left(\frac{\theta}{2}\right) - \frac{2}{3} \cot^3\left(\frac{\theta}{2}\right) [+c]$	M1 A2,1,0	Uses a relevant substitution or integration by parts so as to require a single step to find the integral. PI by the A1 form below If not A2 , award A1 for the form $k \left(\cot\left(\frac{\theta}{2}\right) + \frac{1}{3} \cot^3\left(\frac{\theta}{2}\right) \right) [+c]$ where $k = 2$ or $\frac{1}{2}$ or $-\frac{1}{2}$
		3	

Q	Answer	Marks	Comments
14(c)(iii)	$\text{Area} = \frac{1}{2} \int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{4\pi}{3}\right]} \frac{9}{4} \operatorname{cosec}^4\left(\frac{\theta}{2}\right) [d\theta]$ $= \frac{9}{8} \left[-2 \cot\left(\frac{\theta}{2}\right) - \frac{2}{3} \cot^3\left(\frac{\theta}{2}\right) \right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$ $= \frac{9}{8} \left(\left\{ -2 \cot\left(\frac{2\pi}{3}\right) - \frac{2}{3} \cot^3\left(\frac{2\pi}{3}\right) \right\} - \left\{ -2 \cot\left(\frac{\pi}{6}\right) - \frac{2}{3} \cot^3\left(\frac{\pi}{6}\right) \right\} \right)$ $= \frac{9}{8} \left(\left\{ -2 \left(-\frac{1}{\sqrt{3}}\right) - \frac{2}{3} \left(-\frac{1}{\sqrt{3}}\right)^3 \right\} - \left\{ -2(\sqrt{3}) - \frac{2}{3}(\sqrt{3})^3 \right\} \right)$ $= \frac{9}{8} \left(\left\{ \frac{2}{\sqrt{3}} + \frac{2}{9\sqrt{3}} \right\} - \left\{ -2\sqrt{3} - 2\sqrt{3} \right\} \right)$ $= \frac{16}{3} \sqrt{3}$	<p>M1</p> <p>M1</p> <p>A1</p>	<p>Use of $\frac{1}{2} \int r^2 [d\theta]$ oe with integration attempted.</p> <p>Condone missing or incorrect limits</p> <p>Uses their answer to part (c)(ii) and substitutes their non-zero values for θ found in part (c)(i) as limits with the appropriate subtraction included.</p> <p>PI by the line above in the Answer column followed by the correct final answer.</p> <p>CAO</p>
		3	
	Question 14 Total	15	