OXFORD
INTERNATIONAL
AQA EXAMINATIONS

INTERNATIONAL A-LEVEL
FURTHER MATHEMATICS
FM03

Mark scheme

June 2019

Version: 1.0 Final

jun19FM03/MS

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	$\left(\frac{1}{2r+1}-\frac{1}{2r+3}=\right)\quad \frac{2r+3-(2r+1)}{(2r+1)(2r+3)}$	M1	Condone omission of brackets for the M1 mark
	$=\frac{2}{(2r+1)(2r+3)}$	A1	CSO.
1(b)	Attempt to use method of differences	M1	Must include four terms including two which cancel eg $\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)$ with or without factor of $\frac{1}{\text{c's } k}$
	$(A)\left\{\frac{1}{3}-\frac{1}{2n+3}\right\}$	A1	$\frac{1}{3}-\frac{1}{2n+3}$ after cancellations. Ignore any non-zero multiplier
	$(A)\left\{\frac{2n+3-3}{3(2n+3)}\right\}$	A1	Writing $\frac{1}{3}-\frac{1}{2n+3}$ with a common denominator
	$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}=\frac{1}{2}\left\{\frac{2n}{3(2n+3)}\right\}$ $=\frac{n}{3(2n+3)}$	A1	CSO.
	Total	**6**	

Q	Answer	Marks	Comments
2(a)	$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$	M1	Correct exponential form for $\cosh x$ and attempt to expand
	$= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1$	A1	Correct expansion
	$= \left(\frac{e^{2x} + 2 + e^{-2x}}{2}\right) - 1 = \frac{e^{2x} + e^{-2x}}{2}$ $= \cosh 2x$	A1	CSO, AG
2(b)	$\frac{dy}{dx} = 6\cosh 2x - 5\cosh x + 4$	M1	Differentiates, at least two of three terms correct
	$0 = 6(2\cosh^2 x - 1) - 5\cosh x + 4$	m1	Puts $\frac{dy}{dx} = 0$ and forms a quadratic in $\cosh x$
	$0 = 12\cosh^2 x - 5\cosh x - 2$	A1	Correct quadratic equation in a suitable form for solving
	$0 = (4\cosh x + 1)(3\cosh x - 2)$ $\cosh x = -\frac{1}{4}$, $\cosh x = \frac{2}{3}$	A1	Both values for $\cosh x$ oe
	But $\cosh x \geq 1$, so no (real) solutions	E1	Valid reason(s) for discounting **both** the candidate's two roots.
	(Since) $\frac{dy}{dx} \neq 0$, (the curve has) no stationary points	A1	CSO Previous 5 marks scored and conclusion stated. If '$\frac{dy}{dx} \neq 0$' is missing here, accept statement '$\frac{dy}{dx} = 0$ for stationary points' oe at any stage
	Total	**9**	

Q	Answer	Marks	Comments
3(a)(i)	$\alpha\beta+\beta\gamma+\gamma\alpha=3$	B1	Condone 9/3
3(a)(ii)	$\alpha^2+\beta^2+\gamma^2$ = $(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $\alpha+\beta+\gamma=-\frac{b}{a}=0$ $\alpha^2+\beta^2+\gamma^2=0-2(3)=-6$	M1 B1 A1	Correct formula Seen or used CSO, AG.
3(a)(iii)	$\alpha^2+\beta^2+\gamma^2<0$ so roots not all real. Coefficients (of cubic eqn) are all real so non-real roots occur in conjugate pairs ie 2 non-real and 1 real	E1 E1	oe oe 0/2 if 'Hence' not used.
3(b)(i)	(Complex conjugate) $1-\sqrt{6}\,\mathrm{i}$ is a root 3rd root: $0-(1-\sqrt{6}\,\mathrm{i})-(1+\sqrt{6}\,\mathrm{i})=-2$ $\alpha\beta\gamma=-2(1+6)=-14$	M1 A1 A1	Or equating real and imaginary parts after substituting $1+\sqrt{6}\,\mathrm{i}$ for z in the given cubic equation.
3(b)(ii)	$r=-3\times\alpha\beta\gamma=42$	B1ft	Ft on $-3\times$candidate's (b)(i) answer
	Total	**10**	

Q	Answer	Marks	Comments
4(a)	$\begin{vmatrix} 1 & 3 & c \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$ 1 (2−3) −3 (1−3) + c (1−2) = 0 $\begin{vmatrix} 1 & 3 & c \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 5 - c = 0; \quad c = 5$	M1 A1	Equates $\begin{vmatrix} 1 & 3 & c \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$ to 0 and attempts to solve for c. CSO, AG be convinced
4(b)	$x+3y+5z=9$ **(1)**; $x+2y+3z=6$ **(2)** $x+y+z=d$ **(3)** Eg $(\mathbf{1})-(\mathbf{2})$: $y+2z=3$ $(\mathbf{2})-(\mathbf{3})$: $y+2z=6-d$ $6-d=3$ $d=3$	M1 A1 A1 m1 A1	Eliminating a variable from two eqns, no more than one indep. error Using a different pair of eqns to get another value for $k(\text{previous } y+2z)$ Forming equation in d only, for consistent equations [If y-eliminated: $4d-12=3d-9$ oe] [If z-eliminated: $6d-12=5d-9$ oe] $d=3$
	Award equivalent marks for other approaches to solving the equations.		
	Total	**7**	

Q	Answer	Marks	Comments
5(a)	$\mathrm{f}(k+1)-2\,\mathrm{f}(k)$		
	$=2^{k+3}+3^{2k+3}-2\left(2^{k+2}+3^{2k+1}\right)$	M1	Seen or used
	$=2^{k+3}+3^{2k+3}-2^{k+3}-2\times3^{2k+1}$	A1	Correct expansion of brackets and $2\times2^{k+2}$ = 2^{k+3} used
	$=9\times3^{2k+1}-2\times3^{2k+1}$ $=7\times3^{2k+1}$	A1	Convincingly shown
5(b)	Let $\mathrm{f}(n)=2^{n+2}+3^{2n+1}$ Assume result true for $n=k$, ie assume $\mathrm{f}(k)$ is a multiple of 7 (*) ie $\mathrm{f}(k)=7\times M$, M an integer		
	From **(a)**, $\mathrm{f}(k+1)=2\,\mathrm{f}(k)+7\times3^{2k+1}$	M1	Attempt at f(k + 1)=…. , ft c's integer a
	$\mathrm{f}(k+1)=7\times(2M+3^{2k+1})$ Now $2k+1$ is a positive integer so $\mathrm{f}(k+1)=7\times(2M+N)=7\times W$, integers N and W $\therefore$ if $\mathrm{f}(k)$ is a multiple of 7 then $\mathrm{f}(k+1)$ is a multiple of 7 (**)	A1	Showing that if $\mathrm{f}(k)$ is a multiple of 7 then $\mathrm{f}(k+1)$ is a multiple of 7
	$\mathrm{f}(1)=8+27(=35)=7\times5 \Rightarrow \mathrm{f}(1)$ is a multiple of 7	B1	Must explicitly show that 8 + 27 is a multiple of 7
	Since $\mathrm{f}(1)$ is a multiple of 7, f(2), f(3), … are multiples of 7 by induction, $2^{n+2}+3^{2n+1}$ is a multiple of 7 for all **integers** $n\geqslant1$	E1	Precise conclusion also dep. on previous 3 marks scored and (*) and (**) present. E0 if statement is not precise, eg 'a multiple of 7 for all $n\geqslant1$'
	Total	7	

Q	Answer	Marks	Comments
6(a)	$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=20e^{2x}+18$ PI: $y_{PI} = p+qxe^{2x}$ $y'_{PI} = qe^{2x}+2qxe^{2x}$ $y''_{PI} = 4qe^{2x}+4qxe^{2x}$	M1	$\pm ae^{2x} \pm bxe^{2x}$
	$4qe^{2x}+4qxe^{2x}+qe^{2x}+2qxe^{2x}$ $-6p-6qxe^{2x}$ $(=20e^{2x}+18)$	M1	Substitution into LHS of DE attempted
	$-6p=18$ and $5q=20$	m1	Dep on 2nd M1 only... equating coefficients to form two equations at least one correct. PI by correct values for both p and q
	$p=-3$; $q=4$ $[y_{PI}=-3+4xe^{2x}]$	A2,1,0	A1 if both values correct but xe^{2x} terms in 2nd M1 line do not cancel
6(b)	Aux. eqn. $m^2+m-6=0$ $(m+3)(m-2)=0$	M1	Factorising or using quadratic formula oe on correct aux eqn. PI by correct two values of 'm' seen/used
	$(y_{CF}=)Ae^{-3x}+Be^{2x}$	A1	Correct CF
	$(y_{GS}=)Ae^{-3x}+Be^{2x}-3+4xe^{2x}$	B1ft	c's CF + c's PI but must have exactly two arbitrary constants
	$x=0$, $y=5$ $\Rightarrow 5=A+B-3$	B1ft	Only ft if previous B1ft scored and GS contains exponentials
	$\frac{dy}{dx}=-3Ae^{-3x}+2Be^{2x}+4e^{2x}+8xe^{2x}$ As $x\to-\infty$, $(e^{2x}\to 0)$ and $xe^{2x}\to 0$	E1	Must treat $xe^{2x}\to 0$ separately
	As $x\to-\infty$, $\frac{dy}{dx}\to 0$ so $A=0$	B1	Coefficient of e^{-3x} is 0
	When $A=0$, $5=0+B-3 \Rightarrow B=8$		
	$y=8e^{2x}-3+4xe^{2x}$	A1	$y=8e^{2x}-3+4xe^{2x}$
	Total	**12**	

Q	Answer	Marks	Comments
7(a)	$\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & 7\\ k & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 7\\ 1\end{bmatrix} = \lambda_1\begin{bmatrix}-1\\ 7\\ 1\end{bmatrix}$	M1	$\mathbf{M}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ with $\mathbf{M}\mathbf{v}_1$ attempted oe
	$\begin{bmatrix}-6\\ 42\\ -k+8\end{bmatrix} = \lambda_1\begin{bmatrix}-1\\ 7\\ 1\end{bmatrix}$, $\lambda_1 = 6$	B1	$\lambda_1 = 6$
	$-k+8 = \lambda_1$	m1	Equating c's components to form an equation involving k
	$k = 2$	A1	$k = 2$
7(b)	$\det(\mathbf{M}-\lambda\mathbf{I}) = \begin{vmatrix}1-\lambda & -1 & 2\\ 0 & 5-\lambda & 7\\ k & 1 & 1-\lambda\end{vmatrix}$		
	$= (1-\lambda)\begin{vmatrix}5-\lambda & 7\\ 1 & 1-\lambda\end{vmatrix} + k\begin{vmatrix}-1 & 2\\ 5-\lambda & 7\end{vmatrix}$	M1	oe
	$= (1-\lambda)[(5-\lambda)(1-\lambda)-7] + k[-17+2\lambda]$	A1ft	Correct (unsimplified) expression for expansion of $\det(\mathbf{M}-\lambda\mathbf{I})$; if value for *k* used, ft on c's non-zero value
	Characteristic eqn: $-\lambda^3 + 7\lambda^2 - 36 = 0$	A1	ACF of the characteristic eqn correctly found eg $(\lambda-3)(6-\lambda)(2+\lambda) = 0$
	Eigenvalues are 3, (6) and -2 so -2 is the least eigenvalue	A1	Eigenvalues 3 and -2 and final conclusion; must see a correct equation
7(c)	$\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & 7\\ k & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = -2\begin{bmatrix}x\\ y\\ z\end{bmatrix}$	M1	$\mathbf{M}\mathbf{v} = -2\mathbf{v}$ oe and attempt to get a system of equations
	$3x - y + 2z = 0$; so $3x+3z = 0$ $7y + 7z = 0 \Rightarrow y = -z$ $kx + y + 3z = 0$ so $2x+2z=0$	A1ft	Three correct ft equations with a later substitution resulting in two equations in just two variables
	$x = y = -z$ so an eigenvector is $\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$	A1	An eigenvector in form $\beta\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$, $\beta \neq 0$
7(d)	$-x = \dfrac{y}{7} = z$; or $x = y = -z$	B1	Either one oe. [since Q is 'write down', the case $\lambda = 3$ is not relevant]
	Total	**12**	

Q	Answer	Mark	Comments
8(a)	$\dfrac{dy}{dx}+\dfrac{1}{x(x+1)}y=2x+3$		
	I.F. is exp$\left(\int \dfrac{1}{x(x+1)}(dx)\right)$	M1	Identified and integration attempted
	$\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$	M1	Partial fractions used as far as finding a value for A and a value for B. PI by next line
	exp $\left(\int\left(\dfrac{1}{x}-\dfrac{1}{x+1}\right)(dx)\right)$		
	(I.F.) = $e^{\ln x-\ln(x+1)}$	A1	
	$=\dfrac{x}{x+1}$	A1	AG be convinced
8(b)	$\left(\dfrac{x}{x+1}\right)\dfrac{dy}{dx}+\dfrac{1}{(x+1)^2}y=\dfrac{x(2x+3)}{x+1}$ $\dfrac{d}{dx}\left[y\ \dfrac{x}{x+1}\right]=\dfrac{x(2x+3)}{x+1}$	M1	Multiplies both sides of the DE by $\dfrac{x}{x+1}$ and then identifies the LHS as the derivative of $y\times\text{I.F.}$ PI by next line.
	$\dfrac{yx}{x+1}=\int\dfrac{x(2x+3)}{x+1}\ (dx)$	A1	oe
	$\dfrac{yx}{x+1}=\int\dfrac{2x^2+3x}{x+1}(dx)$		
	$=\int 2x+1-\dfrac{1}{x+1}(dx)$	m1	Division to reach eg $2x+p+\dfrac{q}{x+1}$, where p and q are non-zero integers. PI by next line
	$=x^2+x-\ln(x+1)\ \ (+A)$	A1	Correct integration of $\dfrac{x(2x+3)}{x+1}$, condone absence of '+constant' here
	$\dfrac{y\,x}{x+1}=x^2+x-\ln(x+1)+A$		
	$y=\left(\dfrac{x+1}{x}\right)[x^2+x-\ln(x+1)+A]$	A1	ACF of the GS
	Total	**9**	

Q	Answer	Mark	Comments
9(a)	$\mathbf{r}.\begin{bmatrix}2\\-1\\2\end{bmatrix}=3$ $\quad\mathbf{r}.\begin{bmatrix}2\\2\\1\end{bmatrix}=4$		
	$\begin{bmatrix}2\\-1\\2\end{bmatrix}.\begin{bmatrix}2\\2\\1\end{bmatrix}$	M1	Use of the scalar product on the two normal vectors $\begin{bmatrix}2\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\2\\1\end{bmatrix}$
	= 4	A1	Correct evaluation of scalar product [Equivalent marks for the use of the modulus of the relevant vector product]
	$\cos\theta=\frac{4}{\sqrt{9}\sqrt{9}}=\frac{4}{9}$	B1	Denominator = 9; correct product of moduli
	Acute angle $=\cos^{-1}\left(\frac{4}{9}\right)=63.6^\circ$	A1	CAO must be 63.6
9(b)	eg (0, a, b), $-a+2b=3,\ 2a+b=4$ Solving gives a common pt (0, 1, 2)	M1 A1	Method to find a common point: Any correct common pt. [other likely ones are (2.5, 0, -1); (5/3, 1/3, 0)]
	$\begin{bmatrix}2\\-1\\2\end{bmatrix}\times\begin{bmatrix}2\\2\\1\end{bmatrix}=\begin{bmatrix}-5\\2\\6\end{bmatrix}$	M1	Finding direction vector of the line: eg $\mathbf{n_1}\times\mathbf{n_2}$ or $\mathbf{n}_2\times\mathbf{n}_1$ attempted or by applying a correct method to obtain and use two common points
		A1	A correct direction vector
	$\mathbf{r}=\begin{bmatrix}0\\1\\2\end{bmatrix}+t\begin{bmatrix}-5\\2\\6\end{bmatrix}$	A1	oe ACF with correct notation eg $(\mathbf{r}-(\mathbf{j}+2\mathbf{k}))\times(-5\mathbf{i}+2\mathbf{j}+6\mathbf{k})=\mathbf{0}$
	Total	**9**	

Q	Answer	Mark	Comments
10(a)	$\frac{dx}{dt} = 1-\cos t$;	B1	oe Correct expression for $\frac{dx}{dt}$
	$\frac{dy}{dt} = 2\left(2\sin\frac{t}{2}\right)\left(\frac{1}{2}\cos\frac{t}{2}\right)$	B1	oe Correct expression for $\frac{dy}{dt}$
	$\frac{dx}{dt} = 1-\cos t = 2\sin^2\frac{t}{2}$	M1	$1-\cos t = 2\sin^2\frac{t}{2}$ used at any stage or better
	$\dot{x}^2 + \dot{y}^2 = 4\sin^4\frac{t}{2} + 4\sin^2\frac{t}{2}\cos^2\frac{t}{2} =$ $4\sin^2\frac{t}{2}\ (\sin^2\frac{t}{2} + \cos^2\frac{t}{2}) = 4\sin^2\frac{t}{2}$		
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\sin^2\frac{t}{2}$	A1	CSO, AG
(b)	(SA=) $2\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\sin^2\frac{t}{2}\left(2\sin\frac{t}{2}\right)dt$	B1	Must include the 2π and dt with a correct integrand. Correct limits seen here or used correctly later.
	$= 8\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos^2\frac{t}{2})\left(\sin\frac{t}{2}\right)dt$		
	Subst: Let $u = \cos\frac{t}{2}$; $\frac{du}{dt} = -\frac{1}{2}\sin\frac{t}{2}$	M1	Valid method to integrate $\sin^3\frac{t}{2}$ eg substitution/inspection [PI by answer $\left(a\cos\frac{t}{2} + \frac{b}{3}\cos^3\frac{t}{2}\right)$] and by parts
	$=(8\pi)\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} (1-u^2)(-2)\,du$	A1	Subst….correct integrand and limits By parts….Must reach a stage where only integrals remaining are multiples of $\sin^3\frac{t}{2}$ or better . By inspection…as M1 above with b=2 or better.
	$= (8\pi)\left[(-2)(u-\frac{u^3}{3})\right]_{(\frac{\sqrt{3}}{2})}^{\left(\frac{1}{\sqrt{2}}\right)}$	A1	Integration of $\sin^3\frac{t}{2}$ complete and correct. By parts…the most likely form for integral of $\sin^3\frac{t}{2}$ will be $-\frac{2}{3}\sin^2\frac{t}{2}\cos\frac{t}{2} - \frac{4}{3}\cos\frac{t}{2}$ oe
	$= -16\pi\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right)\right]$	m1	Correct use of correct limits. Award even if limits reversed on integral sign.
	$= \frac{2\pi}{3}\left(9\sqrt{3} - 10\sqrt{2}\right)$	A1	Be convinced as the form of the answer is given
Q	**Total**	**10**	

Q	Answer	Marks	Comments
11(a)(i)	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$ $z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos n\theta - \mathrm{i}\sin n\theta$ $z^n + \dfrac{1}{z^n} = 2\cos n\theta$	M1 E1 A1	Must be shown either as in soln or using $\dfrac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \times \dfrac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta} =$ AG Note: M1E0 A1 is possible eg for those who just quote $z^{-n} = \cos n\theta - \mathrm{i}\sin n\theta$.
11(a)(ii)	$z - \dfrac{1}{z} = 2\mathrm{i}\sin\theta$	B1	
11(b)	$(z - z^{-1})^6 = z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}$ $= z^6 + z^{-6} - 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) - 20$ $((2\mathrm{i}\sin\theta)^6 =)$ $= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$ $(2\mathrm{i}\sin\theta)^6 = 64\mathrm{i}^6\sin^6\theta = -64\sin^6\theta$ $64\sin^6\theta = 20 - 30\cos 2\theta + 12\cos 4\theta - 2\cos 6\theta$	M1 A1 m1 A1 A1	Attempts to find expansion of $(z - z^{-1})^6$ Expansion correct Groups terms so as to use result in **(a)** PI Showing RHS CSO
11(c)	(Area of leaf) = $\dfrac{1}{2}\int_0^{\pi} 4\sin^6\theta\,\mathrm{d}\theta =$ $\dfrac{1}{32}\int_0^{\pi}(20 - 30\cos 2\theta + 12\cos 4\theta - 2\cos 6\theta)\,\mathrm{d}\theta$ $(\int_0^{\pi}\cos n\theta\,\mathrm{d}\theta = [\dfrac{1}{n}\sin n\theta]_0^{\pi} = 0)$ (Area leaf=) $\dfrac{1}{32}[20\theta]_0^{\pi} = \dfrac{20\pi}{32}$ $\left(= \dfrac{5\pi}{8}\right)$ (Area of disc) = $\pi \times 1^2 = \pi$ % area of disc not covered = $\dfrac{3}{8} \times 100 = 37.5\%$	M1 M1 A1ft B1 A1	Use of $\dfrac{1}{2}\int r^2\,(\mathrm{d}\theta)$ or $\int_0^{\frac{\pi}{2}} r^2\,(\mathrm{d}\theta)$ oe Uses **11(b)** with c's values for a,b,c; then integrates correctly or explains/clearly indicates that cos terms when integrated are 0 at each of the limits A correct area for the leaf; ft on candidate's non-zero values for a,b,c. 37.5 OE with no errors seen
	Total	**14**	

Q	Answer	Mark	Comments
12(a)	$\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 \ldots.$	B1	Do not accept series in powers of $2x$
12(bi)	$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$	B1;B1	$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ oe (B1) ; $\frac{1}{1+x^2}$ (B1)
	$\frac{1}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -\frac{2x}{(1+x^2)^2}$	M1 A1	Pr. rule/Quotient rule used appropriately Correct differentiation of prev line oe
	$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{-2x\ y^2}{(1+x^2)^2} = -2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$		
	$(2x-1)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ (*)	A1	CSO, AG
12(bii)	$\ln y = \tan^{-1}x \Rightarrow y = \mathrm{e}^{\tan^{-1}x}$. From McL.		
	when $x=0$, $y'=1$, $y''=1$, $y'''=3!\,p$, $y^{(iv)} = 4!\,q$	B2,1,0	seen or used as appropriate. B1 if two are correct,
	Differentiating (*) wrt *x*: $2(y')^2 + (2x-1)2y'y'' + y'y'' + yy''' = 0$	B1	A correct equation involving y''' oe
	Sub $x=0$ gives $2(1)^2 + (-1)2(1)(1) + (1)(1) + (1)y'''(0) = 0$		
	$\Rightarrow y'''(0) = -1 \quad \Rightarrow p = \frac{y'''(0)}{3!} = -\frac{1}{6}$	B1	AG Be convinced
	Differentiating wrt *x*: $4y'y'' + 4y'y'' + (4x-2)[y'y''' + (y'')^2] +$ $+ y'y''' + (y'')^2 + y'y''' + yy^{(iv)} = 0$	M1	Product rule/quotient rule used appropriately to obtain an equation involving $y^{(iv)}$ oe
	Sub $x=0$ gives $4(1)(1) + 4(1)(1) - 2[(1)(-1) + 1^2] +$ $(1)(-1) + 1^2 + (1)(-1) + (1)y^{(iv)}(0) = 0$ $\Rightarrow y^{(iv)}(0) = -7$		
	From McL series, $q = \frac{y^{(iv)}(0)}{4!} = -\frac{7}{24}$	A1	CSO

Q	Answer	Mark	Comments
	<u>Alternative for final 4 marks</u> From (b)(i), $\left(1+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(1-2x\right)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$		
	$2x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x^2\right)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2\dfrac{\mathrm{d}y}{\mathrm{d}x}+\left(1-2x\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$	(B1)	
	Sub $x=0$, $3!\,p=-2+1\Rightarrow p=-\dfrac{1}{6}$	(B1)	
	$2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+(1+x^2)\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$ $=-4\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1-2x\right)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$	(M1)	
	Sub $x=0$, $2+4!\,q=-4-1\Rightarrow q=-\dfrac{7}{24}$	(A1)	CSO
12(c)	$\lim\limits_{x\to0}\left[\dfrac{\mathrm{e}^{\tan^{-1}x}-\mathrm{e}^x}{2x-\sin 2x}\right]=$		
	$=\lim\limits_{x\to0}\dfrac{-\dfrac{x^3}{6}+qx^4-\dfrac{x^3}{6}-\dfrac{x^4}{24}....}{2x-2x+\dfrac{4x^3}{3}+O(x^5)}$	M1	Substitution of series
	$=\lim\limits_{x\to0}\dfrac{-\dfrac{2}{6}+O(x)}{\dfrac{4}{3}+O(x^2)}$, (so limit exists)	m1	Dividing numerator and denominator by x^3 to get $\lim\limits_{x\to0}\dfrac{a+O(x)}{b+O(x^2)}$, so limit exists $= a/b$. In place of $O(\ )$ may have equivalent term(s)
	$=-0.25$	A1	Correct value for the limit. (A0 if previous 2 marks not scored)
	Total	**15**	