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(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	Reflection in [the plane] $x = y$	**M1** **A1**	Reflection $x = y$ **oe** **M0** If more than one transformation **SC1** for ‘reflect’ or ‘reflected’ and $x = y$
		2	

Q	Answer	Marks	Comments
1(b)	y–axis	**B1**	**oe** eg $x = z = 0$
		1	

	Question 1 Total	**3**	

Q	Answer	Marks	Comments
2	$8\mathbf{u}\times\mathbf{u}-16\mathbf{u}\times\mathbf{v}+12\mathbf{u}\times\mathbf{w}$ $+6\mathbf{v}\times\mathbf{u}-12\mathbf{v}\times\mathbf{v}+9\mathbf{v}\times\mathbf{w}$ $+12\mathbf{w}\times\mathbf{u}-24\mathbf{w}\times\mathbf{v}+18\mathbf{w}\times\mathbf{w}$	**M1**	Expansion of brackets at least five correct **PI** Condone lack of $\times$ or $\wedge$ except for the final mark but check order of letters in each term.
	$=-16\mathbf{u}\times\mathbf{v}+12\mathbf{u}\times\mathbf{w}+6\mathbf{v}\times\mathbf{u}+9\mathbf{v}\times\mathbf{w}$ $+12\mathbf{w}\times\mathbf{u}-24\mathbf{w}\times\mathbf{v}$	**M1**	$\mathbf{u}\times\mathbf{u}=\mathbf{v}\times\mathbf{v}=\mathbf{w}\times\mathbf{w}=\mathbf{0}$ stated or used
	$=-16\mathbf{u}\times\mathbf{v}-12\mathbf{w}\times\mathbf{u}-6\mathbf{u}\times\mathbf{v}+9\mathbf{v}\times\mathbf{w}$ $+12\mathbf{w}\times\mathbf{u}+24\mathbf{v}\times\mathbf{w}$ $=-22\mathbf{u}\times\mathbf{v}+33\mathbf{v}\times\mathbf{w}$	**M1**	Clear evidence of the consistent use of any two of: $\mathbf{u}\times\mathbf{w}=-\mathbf{w}\times\mathbf{u}$ $\mathbf{w}\times\mathbf{v}=-\mathbf{v}\times\mathbf{w}$ $\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}$
	$=165\mathbf{i}-44\mathbf{j}$	**B1**	Either $165\mathbf{i}$ or $-44\mathbf{j}$
		A1CSO	$165\mathbf{i}-44\mathbf{j}$ with no other terms and correct notation used throughout
		5	

	Question 2 Total	**5**	

Q	Answer	Marks	Comments
3	When $n = 1$, $u_1 = \frac{5+1}{5-3} = \frac{6}{2} = 3$	**B1**	Correct values to show formula true for $n = 1$
	Assume formula true for $n = k$ **(*)**, integer $k \geq 1$, so $u_{k+1} = \frac{9\left(\frac{5k+1}{5k-3}\right)-5}{5\left(\frac{5k+1}{5k-3}\right)-1}$	**M1**	Assumes formula true for $n = k$ and considers $u_{k+1} = \frac{9\left(\frac{5k+1}{5k-3}\right)-5}{5\left(\frac{5k+1}{5k-3}\right)-1}$ **oe**
	$u_{k+1} = \frac{9(5k+1)-5(5k-3)}{5(5k+1)-1(5k-3)}$	**m1**	Simplifies into a single fraction
	$u_{k+1} = \frac{20k+24}{20k+8}$	**A1**	
	$u_{k+1} = \frac{5k+6}{5k+2}$		
	$u_{k+1} = \frac{5(k+1)+1}{5(k+1)-3}$	**A1**	Be convinced
	Hence formula is true for $n = k + 1$ **(**)** and since true for $n = 1$ **(***)**, formula is true for $n = 1, 2, 3, \ldots$ by induction **(****)**	**E1**	Must have **(*)** , **(**)**, **(***)**, present, previous 5 marks scored and a final statement **(****)** clearly indicating that it relates to positive integers
		6	

	Question 3 Total	**6**	

Q	Answer	Marks	Comments
4	Aux equation $m^2+3m+2=0 \qquad m=\dfrac{-3\pm\sqrt{9-8}}{2}$	M1	Using quadratic formula **oe** on correct aux. equation. **PI** by correct values of m seen/used
	$[y_{CF}=]\quad A\mathrm{e}^{-x}+B\mathrm{e}^{-2x}$	A1	Correct CF
	$y_{PI}=a\sin 2x+b\cos 2x$	M1	**PI** by correct y_{PI}
	$(-4a-6b+2a)\sin 2x+(-4b+6a+2b)\cos 2x$ $=2\sin 2x+14\cos 2x$	m1	y'_{PI} and y''_{PI} both of the form $p\sin 2x+q\cos 2x$ and substitution into the DE to form an equation in x **PI**
	$-2a-6b=2 \qquad 6a-2b=14$	A1	Seen or used, **PI** by $a=2$, $b=-1$
	$y_{PI}=2\sin 2x-\cos 2x$	A1	Correct y_{PI} seen/used
	$[y_{GS}=]\quad A\mathrm{e}^{-x}+B\mathrm{e}^{-2x}+2\sin 2x-\cos 2x$	B1ft	c's CF + c's PI but must have exactly two arbitrary constants
		7	

	Question 4 Total	**7**	

Q	Answer	Marks	Comments
5(a)	$\sin(2r+1)x-\sin(2r-1)x=2\cos\frac{4rx}{2}\sin\frac{2x}{2}$ $\frac{1}{2}\left[\sin(2r+1)x-\sin(2r-1)x\right]=\cos 2rx\sin x$	B1	Replaces A by $(2r+1)x$ and B by $(2r-1)x$ and obtains $\cos 2rx \sin x$
		1	

Q	Answer	Marks	Comments
5(b)	$\sum_{r=1}^{n}\cos 2rx\sin x$ $=\sum_{r=1}^{n}\frac{1}{2}\left[\sin(2r+1)x-\sin(2r-1)x\right]$	M1	Uses result in **part (a)**
	$\sum_{r=1}^{n}\sin x\left(1-2\sin^2 rx\right)$	M1	Uses $\cos 2rx=\pm1\pm2\sin^2 rx$
	$=\frac{1}{2}\begin{bmatrix}\sin 3x-\sin x+\sin 5x-\sin 3x+\ldots\\+\sin(2n-1)x-\sin(2n-3)x\\+\sin(2n+1)x-\sin(2n-1)x\end{bmatrix}$	M1	Uses method of differences showing at least terms that cancel
	$\sin x\left(n-2\sum_{r=1}^{n}\sin^2 rx\right)$	B1	$\sum_{r=1}^{n}1=n$ seen/used at any stage
	$=\frac{1}{2}\left[\sin(2n+1)x-\sin x\right]$	A1	$\sin(2n+1)x-\sin x$
	$\sum_{r=1}^{n}\sin^2 rx=\frac{n}{2}-\frac{1}{4\sin x}\left(\sin(2n+1)x-\sin x\right)$ $=\frac{n}{2}-\frac{1}{4\sin x}\left(2\cos\left[\frac{(2n+2)x}{2}\right]\sin\left(\frac{2nx}{2}\right)\right)$ $\sum_{r=1}^{n}\sin^2 rx=\frac{n}{2}-\frac{\sin nx\cos(n+1)x}{2\sin x}$	A1	**AG** Printed result convincingly obtained
		6	

	Question 5 Total	**7**	

Q	Answer	Marks	Comments
6	$u = x^2 + 1 \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x$ $\mathrm{d}v = \mathrm{e}^{-x}\,\mathrm{d}x \Rightarrow v = -\mathrm{e}^{-x}$ $\int (x^2+1)\mathrm{e}^{-x}\,\mathrm{d}x$ $= (x^2+1)(-\mathrm{e}^{-x}) - \int -\mathrm{e}^{-x}2x\,\mathrm{d}x$	B1	
	$= -(x^2+1)\,\mathrm{e}^{-x} + 2\left(-x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\,\mathrm{d}x\right)$	M1	Applies integration by parts twice
	$= -(x^2+1)\,\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} \quad [+c]$	A1	Correct integration of $(x^2+1)\mathrm{e}^{-x}$
	$\mathrm{I} = \int_0^{\infty} (x^2+1)\mathrm{e}^{-x}\,\mathrm{d}x$ $= \lim_{a\to\infty} \int_0^{a} (x^2+1)\mathrm{e}^{-x}\,\mathrm{d}x$	M1	Evidence of limit ∞ replaced by a (**oe**) $\lim_{a\to\infty}$ seen at any stage with no remaining lim relating to 0
	$= \lim_{a\to\infty}\left(-a^2\,\mathrm{e}^{-a} - 2a\,\mathrm{e}^{-a} - 3\,\mathrm{e}^{-a}\right) - (-3)$	M1	$\mathrm{F}(a) - \mathrm{F}(0)$
		A1	**ACF**
	$\lim_{a\to\infty}\left(a^n\mathrm{e}^{-a}\right) = 0$	E1	Stated in general form or for $n = 1$ or for $n = 2$
	$\int_0^{\infty} (x^2+1)\mathrm{e}^{-x}\,\mathrm{d}x = 3$	A1	First 6 marks must have been scored but can be awarded even if previous **E1** not scored
		8	

	Question 6 Total	**8**	

Q	Answer	Marks	Comments
7(a)	I.F. is $e^{\int\left(x^{-1}-(x+2)^{-1}\right)dx} = e^{\ln x - \ln(x+2)}$	M1	I.F. identified and integration attempted
		A1	Correct integration
	I.F. $=\dfrac{x}{x+2}$	A1	Correct integrating factor
	$\dfrac{x}{x+2}y=\int\dfrac{x^2}{x+2}\,dx$	M1	Multiplying both sides of the given DE by their I.F. and integrating LHS to get $y\times$I.F. RHS must not remain as x
	$\dfrac{x}{x+2}y=\int x-2+\dfrac{4}{x+2}\,dx$	M1	Expressing $\dfrac{x^2}{x+2}$ in the form $x+a+\dfrac{b}{x+2}$ **PI**
	$\dfrac{x}{x+2}y=\dfrac{x^2}{2}-2x+4\ln(x+2)\quad[+A]$	A1ft	Correct integration of $x+a+\dfrac{b}{x+2}$ **ft** on candidate's non-zero numerical values for a and b; condone absence of constant of integration
	$\dfrac{x}{x+2}y=\dfrac{x^2}{2}-2x+4\ln(x+2)+A$	A1	Correct GS. Apply **ISW** if incorrect further rearrangement
		7	

Q	Answer	Marks	Comments
7(b)	Substituting $\dfrac{dy}{dx}=0$ and $x = 2$ into the DE $0+\left(\dfrac{1}{2}-\dfrac{1}{4}\right)y=2\quad\Rightarrow y=8$	M1	Uses $\dfrac{dy}{dx}=0$ when $x = 2$ to obtain either a non-zero value for y or a value for the constant in their GS
	$\dfrac{2}{4}\times 8=2-4+4\ln(4)+A\ \Rightarrow A=6-4\ln(4)$	m1	Obtains a value for their GS constant in the form $p+q\ln(4)$
	$y=\dfrac{x+2}{x}\left(\dfrac{x^2}{2}-2x+4\ln(x+2)+6-4\ln 4\right)$	A1	$y=f(x)$ with **ACF** for $f(x)$
		3	

	Question 7 Total	**10**	

Q	Answer	Marks	Comments
8(a)	[distance =] $\dfrac{14}{\sqrt{1^2+1^2+1^2}}$	M1	**PI**
	$=\dfrac{14}{\sqrt{3}}$	A1	**ACF** but must be exact eg $\dfrac{14\sqrt{3}}{3}$
		2	

Q	Answer	Marks	Comments
8(b)(i)	General point on *L* $\left(3\lambda+2,\ 2\lambda-1,\ \dfrac{\lambda+4}{2}\right)$	B1	**ACF**
	$\begin{bmatrix}3\lambda+2\\2\lambda-1\\\frac{1}{2}\lambda+2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}=14$ $\Rightarrow(3\lambda+2)+(2\lambda-1)+\dfrac{\lambda}{2}+2=14$	M1	Substitutes general point for $\mathbf{r}$ in the equation of Π_1 and evaluates the scalar product to form an equation for λ **oe**
	$\Rightarrow\dfrac{11}{2}\lambda+3=14\ \ \Rightarrow\lambda=2\Rightarrow P(8,\ 3,\ 3)$	A1	$(8,\ 3,\ 3)$ Condone $\begin{bmatrix}8\\3\\3\end{bmatrix}$
		3	

Q	Answer	Marks	Comments
8(b)(ii)	Direction vector for *L*: $\mathbf{d}=\begin{bmatrix}3\\2\\0.5\end{bmatrix}$	B1	Seen or used
	$\mathbf{n}\cdot\mathbf{d}=3+2+0.5$	M1	Finds a numerical expression for the scalar product of the normal to Π_1 and $\mathbf{d}$; **ft** one error in $\mathbf{d}$
	$\sqrt{1^2+1^2+1^2}\,\sqrt{3^2+2^2+0.5^2}\cos\theta=5.5$	m1	**ft** one error in $\mathbf{d}$
	Angle = $90^\circ-\cos^{-1}\left(\dfrac{11}{\sqrt{159}}\right)=60.7^\circ$	A1	**CAO** 60.7° (Condone missing $^\circ$)
		4	

Q	Answer	Marks	Comments
8(c)	$\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix}$		or $\begin{bmatrix}0\\1\\-1\end{bmatrix} \times \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$
		M1	Vector product of the two normal vectors, in either order, attempted
		A1	Correct evaluation of the vector product
	[Direction cosines] $\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$		or $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$
		A1ft	**oe** exact form **ft** on incorrect evaluation of the vector product
		3	

	Question 8 Total	**12**	

	Answer	Marks	Comments
9(a)	det **M** $= 4(12-k-1)-3(15-k-1)+k(5-4)$ $= 44-4k-42+3k+k$	**M1**	Correct method to expand det **M**
	det **M** $=2\neq 0$ so **M** is a non-singular matrix	**A1**	det **M** $=2\neq 0$ with conclusion
		2	

Q	Answer	Marks	Comments
9(b)	Cofactor matrix		
	$\begin{bmatrix} 11-k & k-14 & 1 \\ k-9 & 12-k & -1 \\ 3-k & k-4 & 1 \end{bmatrix}$	**M1**	One complete row or column correct
		A2	All nine entries correct else **A1** for at least six entries correct
	Inverse matrix $\mathbf{M}^{-1}$ =	**M1**	Transpose of their cofactors with no more than one further error **and** division by their det **M** $(\neq 0)$
	$\frac{1}{2}\begin{bmatrix} 11-k & k-9 & 3-k \\ k-14 & 12-k & k-4 \\ 1 & -1 & 1 \end{bmatrix}$	**A1**	Correct $\mathbf{M}^{-1}$ scores 5 marks
		5	

Q	Answer	Marks	Comments
9(c)	When $k = 5$, $\mathbf{M}^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4.5 & 3.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$	M1	Substitutes $k = 5$ into their $\mathbf{M}^{-1}$ from **(b)** or correct $\mathbf{M}^{-1}$ from a calculator
	$\begin{bmatrix} 3 & -2 & -1 \\ -4.5 & 3.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \lambda \\ -1+\lambda \\ 4-2\lambda \end{bmatrix}$	M1	Product of candidate's $\mathbf{M}^{-1}$ when $k = 5$ and RHS of vector equation for L. Condone one miscopy.
	$= \begin{bmatrix} 3\lambda+2-2\lambda-4+2\lambda \\ -4.5\lambda-3.5+3.5\lambda+2-\lambda \\ 0.5\lambda+0.5-0.5\lambda+2-\lambda \end{bmatrix}$	A1ft	At least two of the three components correct; accept unsimplified versions
	$= \begin{bmatrix} 3\lambda-2 \\ -2\lambda-1.5 \\ -\lambda+2.5 \end{bmatrix}$		
	$\left(\mathbf{r} - \begin{bmatrix} -2 \\ -1.5 \\ 2.5 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \mathbf{0}$	m1	**ft** on candidate's three components each of the form $a\lambda + b$ with no more than one zero value for the numerical constants a, b.
		A1	Condone r and/or 0 not underlined
		5	

	Question 9 Total	**12**	

Q	Answer	Marks	Comments
10(a)	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ...$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - ...$	B1	Series expansion for $\ln(1-x)$ at least up to x^3
	$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $= \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$	M1	
	$\tanh^{-1}x = \frac{1}{2}\left(2x + \frac{2x^3}{3} + \frac{2x^5}{5} + ...\right)$ $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + ...$	A1	**AG** printed answer obtained convincingly with no relevant terms missing in other series above.
		3	

Q	Answer	Marks	Comments
10(b)(i)	$y = \tan x$; $\frac{dy}{dx} = \sec^2 x$; $\frac{d^2y}{dx^2} = 2\sec^2 x \tan x = 2\tan^3 x + 2\tan x$	B1	**ACF** Correct first two derivatives eg $y' = 1 + y^2$; $y'' = 2y\,y'$
	$\frac{d^3y}{dx^3} = 2\sec^2 x + 6\tan^2 x \sec^2 x$ $\frac{d^3y}{dx^3} = 6\tan^4 x + 8\tan^2 x + 2$	B1	**ACF** Correct third derivative eg $y''' = 2(y')^2 + 2y\,y''$
	$\frac{d^4y}{dx^4} = 24\tan^3 x \sec^2 x + 16\tan x \sec^2 x$ $\frac{d^4y}{dx^4} = 24\tan^5 x + 40\tan^3 x + 16\tan x$	M1	Correct method to differentiate two powers of $\tan x$ or two powers of $\sec x$ or a term of the form $\tan^2 x \sec^2 x$ **oe** or both $(y')^2$ and $y\,y''$ terms when finding the fourth derivative eg $y^{(iv)} = 6y'y'' + 2y\,y'''$
	$\frac{d^5y}{dx^5} =$ $120\tan^4 x \sec^2 x + 120\tan^2 x \sec^2 x + 16\sec^2 x$ $\Rightarrow \frac{d^5y}{dx^5} = 16$ when $x = 0$	A1	eg $y^{(v)} = 6(y'')^2 + 8y'y''' + 2y\,y^{(iv)}$ **AG** Printed answer obtained convincingly.
		4	

Q	Answer	Marks	Comments
10(b)(ii)	From **part (b)(i)** $y(0)=0;\ y'(0)=1;\ y''(0)=0;$ $y'''(0)=2;\ y^{(\text{iv})}(0)=0;\ y^{(\text{v})}(0)=16$	**B1**	All six stated/used with at least four correct. Accept if seen in **(b)(i)**
	$\tan x = 0 + x + \frac{0x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{16x^5}{5!} + \ldots$	**M1**	Maclaurin series applied for the candidate's at least three non-zero values
	$\tanh^{-1}x - \tan x =$ $x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots - \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\right)$		
	$\tanh^{-1}x - \tan x = \frac{x^5}{15} + \ldots$	**A1**	**AG** Printed answer obtained convincingly
		3	

Q	Answer	Marks	Comments
10(c)	$\cos 2x = 1 - \frac{4x^2}{2} + O(x^4)$	B1	$\cos 2x = 1 - \frac{4x^2}{2} + \ldots$ **oe** seen or used
	$\left[\frac{\tan x + \tanh^{-1}x - 2x}{x(1-\cos 2x)}\right]$ $= \frac{2x + \frac{2}{3}x^3 + \frac{1}{3}x^5 + \ldots - 2x}{x\left(1 - 1 + \frac{4}{2}x^2 + O(x^4)\right)}$	M1	Substitutes series expansions
	$\lim_{x \to 0}\left[\frac{\tan x + \tanh^{-1}x - 2x}{x(1-\cos 2x)}\right]$ $= \lim_{x \to 0}\left[\frac{\frac{2}{3}x^3 + O(x^5)}{\frac{4}{2}x^3 + O(x^5)}\right]$ $= \lim_{x \to 0}\left[\frac{\frac{2}{3} + O(x^2)}{\frac{4}{2} + O(x^2)}\right]$ [so the limit exists]	m1	Divides numerator and denominator by x^3 to reach the form $\lim_{x \to 0}\left[\frac{P + O(x^2)}{Q + O(x^2)}\right]$, so limit exists $= \frac{P}{Q}$ In place of $O(\ )$ we may see equivalent term(s)
	$= \frac{1}{3}$	A1	$= \frac{1}{3}$
		4	

	Question 10 Total	**14**	

Q	Answer	Marks	Comments
11(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 2$	**M1**	Both attempted, at least one correct **oe** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x}}$
	$S = 2\pi\int_0^{\sqrt{3}} 2t\sqrt{4t^2+4}\ \mathrm{d}t$	**M1**	**oe** Substitutes into a correct formula for the surface area S. Condone incorrect/absent limits and absent dt
		A1	Correct definite integral
	$S = 8\pi\left[\frac{1}{3}\left(t^2+1\right)^{1.5}\right]_0^{\sqrt{3}}$	**M1**	Integrates $\lambda t\sqrt{t^2+1}$ or $\beta\sqrt{x+1}$ correctly
	$S = \frac{8\pi}{3}(8-1) = \frac{56\pi}{3}$	**A1**	**AG** Printed answer obtained convincingly
		5	

Q	Answer	Marks	Comments
11(b)	$s = \int_0^{\sqrt{3}} \sqrt{4t^2+4}\ \mathrm{d}t = 2\int_0^{\sqrt{3}} \sqrt{t^2+1}\ \mathrm{d}t$	**M1**	**oe** Substitutes into a correct formula for the arc length s. **ft** derivatives from **part (a)** if incorrect. Condone incorrect/absent limits and absent dt
	$\int \sqrt{t^2+1}\ \mathrm{d}t = t\sqrt{t^2+1} - \int t\frac{t}{\sqrt{t^2+1}}\ \mathrm{d}t$	**M1**	Uses integration by parts with $u = \sqrt{t^2+1}$ and $\mathrm{d}v = \mathrm{d}t$
		A1	$\int \sqrt{t^2+1}\ \mathrm{d}t = t\sqrt{t^2+1} - \int t\frac{t}{\sqrt{t^2+1}}\ \mathrm{d}t$
	$\int \sqrt{t^2+1}\ \mathrm{d}t = t\sqrt{t^2+1} - \int \sqrt{t^2+1} - \frac{1}{\sqrt{t^2+1}}\ \mathrm{d}t$	**m1**	Uses $\frac{t^2}{\sqrt{t^2+1}} = \sqrt{t^2+1} - \frac{1}{\sqrt{t^2+1}}$
	$\int \sqrt{t^2+1}\ \mathrm{d}t = t\sqrt{t^2+1} + \sinh^{-1}t - \int \sqrt{t^2+1}\ \mathrm{d}t$	**A1**	
	$2\int_0^{\sqrt{3}} \sqrt{t^2+1}\ \mathrm{d}t = \left[t\sqrt{t^2+1} + \sinh^{-1}t\right]_0^{\sqrt{3}}$	**A1**	
	[Arc length OP] $= 2\sqrt{3} + \sinh^{-1}\left(\sqrt{3}\right)$	**A1**	**AG** Printed answer obtained convincingly
		7	

Q	Answer	Marks	Comments
11(b) ALT	$s=\int_0^{\sqrt{3}}\sqrt{4t^2+4}\ \mathrm{d}t = 2\int_0^{\sqrt{3}}\sqrt{t^2+1}\ \mathrm{d}t$	**M1**	**oe** Substitutes into a correct formula for the arc length s. **ft** derivatives from **part (a)** if incorrect. Condone incorrect/absent limits and absent dt
	Let $t=\sinh u \Rightarrow$ $s=2\int_0^{\sinh^{-1}\sqrt{3}}\sqrt{\sinh^2 u+1}\cosh u\ \mathrm{d}u$	**M1**	Uses a suitable substitution to integrate. Ignore limits
	$s=2\int_0^{\sinh^{-1}\sqrt{3}}\cosh^2 u\ \mathrm{d}u$	**A1**	Ignore limits
	$s=\int_0^{\sinh^{-1}\sqrt{3}}(1+\cosh 2u)\,\mathrm{d}u$	**m1**	Ignore limits
	$s=\left[u+\frac{1}{2}\sinh 2u\right]_0^{\sinh^{-1}\sqrt{3}}=$	**A1**	Correct integration with correct limits
	$\sinh^{-1}\sqrt{3}+\frac{2}{2}\sinh\left(\sinh^{-1}\sqrt{3}\right)\cosh\left(\sinh^{-1}\sqrt{3}\right)$ $=\sinh^{-1}\left(\sqrt{3}\right)+\sqrt{3}\sqrt{1+\left(\sqrt{3}\right)^2}$	**A1**	**PI** by the next line
	[Arc length OP] $=2\sqrt{3}+\sinh^{-1}\left(\sqrt{3}\right)$	**A1**	**AG** Printed answer obtained convincingly
		7	

	Question 11 Total	**12**	

Q	Answer	Marks	Comments
12(a)(i)	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$	**M1**	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$
	$z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos(n\theta) - \mathrm{i}\sin(n\theta)$	**B1**	Need both lines or multiplies both numerator and denominator of $\frac{1}{z^n}$ by $\cos(n\theta) - \mathrm{i}\sin(n\theta)$
	$z^n + \frac{1}{z^n} = 2\cos n\theta$	**A1**	**AG** [**M1B0A1** is possible eg for those who just quote the result $z^{-n} = \cos(n\theta) - \mathrm{i}\sin(n\theta)$]
		3	

Q	Answer	Marks	Comments
12(a)(ii)	$\left(-64\sin^6\theta\right)\left(4\cos^2\theta\right) = \left(z - \frac{1}{z}\right)^4 \left(z^2 - \frac{1}{z^2}\right)^2$ $= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^4 - 2 + \frac{1}{z^4}\right)$	**M1 A1**	**M1** for expansions of both brackets on RHS attempted with at least one correct. **A1** for all four brackets raised to powers expanded correctly
	$-256\sin^6\theta\cos^2\theta = z^8 + \frac{1}{z^8} - 4\left(z^6 + \frac{1}{z^6}\right) +$ $+4\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) - 10$ $= 2\cos 8\theta - 8\cos 6\theta + 8\cos 4\theta + 8\cos 2\theta - 10$	**A1**	RHS of previous line multiplied out and result in **part (a)(i)** used
	$128\sin^6\theta\cos^2\theta$ $= 5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta$	**A1**	**AG** Printed answer obtained convincingly
		4	

Q	Answer	Marks	Comments
12(b)(i)	$r = 32\sin^3\theta\cos\theta$ $\dfrac{dr}{d\theta} = 96\sin^2\theta\cos^2\theta - 32\sin^4\theta$	M1	$\dfrac{dr}{d\theta} = \pm a\sin^2\theta\cos^2\theta \pm b\sin^4\theta$
	At P $32\sin^2\theta\left(3\cos^2\theta - \sin^2\theta\right) = 0$ $\Rightarrow \tan\theta = \sqrt{3}$	m1	Solves $\dfrac{dr}{d\theta} = 0$ using a valid method to obtain a positive value for either $\tan\theta$ or $\cos\theta$ or $\sin\theta$ **PI** if $\theta = \dfrac{\pi}{3}$
	$\theta = \dfrac{\pi}{3}$	A1	$\theta = \dfrac{\pi}{3}$
	$\Rightarrow r = 6\sqrt{3}$ $\Rightarrow P\left(6\sqrt{3}, \dfrac{\pi}{3}\right)$	A1	$r = 6\sqrt{3}$
		4	

Q	Answer	Marks	Comments
12(b)(ii)	Area $= \dfrac{1}{2}\int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{\pi}{2}\right]} (32\sin^3\theta\cos\theta)^2 [d\theta]$	M1	Use of $\dfrac{1}{2}\int r^2 [d\theta]$
	$= 4\int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{\pi}{2}\right]} 128\sin^6\theta\cos^2\theta [d\theta]$		
	$= 4\int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{\pi}{2}\right]} \left\{\begin{matrix} 5 - 4\cos 2\theta - 4\cos 4\theta \\ +4\cos 6\theta - \cos 8\theta \end{matrix}\right\} [d\theta]$	M1	Uses the printed result from **part (a)(ii)**
	$= 4\left[\begin{matrix} 5\theta - 2\sin 2\theta - \sin 4\theta \\ +\dfrac{2}{3}\sin 6\theta - \dfrac{1}{8}\sin 8\theta \end{matrix}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	A1	Correct integration
	$= 4\left[\dfrac{5\pi}{2} - \left(\dfrac{5\pi}{3} - \sqrt{3} + \dfrac{\sqrt{3}}{2} + 0 - \dfrac{\sqrt{3}}{16}\right)\right]$		
	$= \dfrac{20\pi}{6} + 4\left(\dfrac{16\sqrt{3} - 8\sqrt{3} + \sqrt{3}}{16}\right) = \dfrac{10\pi}{3} + \dfrac{9\sqrt{3}}{4}$	A1	Correct answer in the requested form $a\pi + b\sqrt{n}$
		4	

	Question 12 Total	**15**	

Q	Answer	Marks	Comments
13(a)	$tx^3 + ux^2 + vx + w = 0$ Let roots be $a-d$, a , $a+d$ so $a-d+a+a+d = -\frac{u}{t} \Rightarrow a = -\frac{u}{3t}$	**M1**	$\alpha+\beta+\gamma = -\frac{u}{t}$ used
	$ta^3 + ua^2 + va + w = 0$	**M1**	Substitutes their middle root for x in the given cubic equation or substitutes $u=-3at$, $v=t\left(3a^2-d^2\right)$ and $w=-at\left(a^2-d^2\right)$ into the LHS of the equation in 13(a)
	$t\left(\frac{-u}{3t}\right)^3 + u\left(\frac{-u}{3t}\right)^2 + v\left(\frac{-u}{3t}\right) + w = 0$ $-u^3 + 3u^3 - 9tuv + 27t^2w = 0$ $2u^3 - 9tuv + 27t^2w = 0$	**A1**	**AG** Printed result convincingly obtained
		3	

Q	Answer	Marks	Comments
13(a) ALT	$tx^3 + ux^2 + vx + w = 0$ Let roots be a , $a+d$, $a+2d$, so $a + a + d + a + 2d = -\frac{u}{t} \Rightarrow 3(a+d) = -\frac{u}{t}$	**M1**	$\alpha+\beta+\gamma = -\frac{u}{t}$ used
	$a(a+d)(a+2d) = -\frac{w}{t} \Rightarrow a(a+2d) = \frac{3w}{u}$ $a(a+d)+(a+d)(a+2d)+a(a+2d) = \frac{v}{t}$ $2(a+d)^2 + a(a+2d) = \frac{v}{t}$	**M1**	$\alpha\beta\gamma = -\frac{w}{t}$ and $\sum\alpha\beta = \frac{v}{t}$ used to form an equation which requires just substitution of known expressions and simplification to reach the printed answer
	$\frac{2}{9}\left(\frac{-u}{t}\right)^2 + \frac{3w}{u} = \frac{v}{t}$ $2u^3 + 27t^2w = 9tvu$ $2u^3 - 9tuv + 27t^2w = 0$	**A1**	**AG** Printed result convincingly obtained
		3	

Q	Answer	Marks	Comments
13(b)(i)	Comparing $kx^3 - 36x^2 + mx - 3 = 0$ with $tx^3 + ux^2 + vx + w = 0$ gives $t = k$, $u = -36$, $v = m$, $w = -3$ $\alpha + \beta + \gamma = \frac{36}{k} \Rightarrow a = \frac{12}{k}$ $\alpha\beta\gamma = \frac{3}{k} \Rightarrow a\left(a^2 - d^2\right) = \frac{3}{k}$ $\Rightarrow \frac{12}{k}\left(\frac{144}{k^2} - d^2\right) = \frac{3}{k}$	**M1**	$\alpha\beta\gamma = \frac{3}{k}$ and $\alpha + \beta + \gamma = \frac{36}{k}$ both used to form an equation in d and k only
	$\Rightarrow d^2 = \frac{576 - k^2}{4k^2}$	**A1**	**ACF** eg $d^2 = \frac{144}{k^2} - \frac{1}{4}$
		2	

Q	Answer	Marks	Comments
13(b)(ii)	Comparing $kx^3 - 36x^2 + 38x - 3 = 0$ with $tx^3 + ux^2 + vx + w = 0$ gives $t = k$, $u = -36$, $v = m$, $w = -3$ so $2(-36)^3 - 9k(38)(-36) + 27k^2(-3) = 0$ $\Rightarrow k^2 - 152k + 1152 = 0$	**M1**	Using the result in **part (a)** to form a quadratic equation in k
	$\Rightarrow k = 8$, $k = 144$	**A1**	Both correct values for k
	When $k = 8$, $d = \pm\sqrt{2}$ When $k = 144$, $d = \pm\sqrt{\frac{35}{144}}$ i	**A2,1**	**oe** **A2** all four correct exact values for d otherwise **A1** for any two correct
		4	

	Question 13 Total	**9**	