Pearson Edexcel

Mark Scheme (Results)

NOTE:
This is the original version of WFM03_01 Mark Scheme, used by examiners.
Question 5c cannot be answered, as **A** is not a symmetric matrix.

Appropriate actions have been taken to ensure candidates who sat this version have not been impacted by this error.

Please do not use this version of the Mark Scheme for mocks and future student assessments. An updated version of the Mark Scheme will be released to be used for mocks and class assessments

January 2023

Pearson Edexcel International Advanced Level
In Further Pure Mathematics F3 (WFM03)
Paper 01

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

PEARSON EDEXCEL IAL MATHEMATICS

General Instructions for Marking

1. The total number of marks for this paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

<u>‘M’ marks</u>
These are marks given for a correct method or an attempt at a correct method. In Mechanics they are usually awarded for the application of some mechanical principle to produce an equation.
e.g. resolving in a particular direction, taking moments about a point, applying a suvat equation, applying the conservation of momentum principle etc.
The following criteria are usually applied to the equation.

To earn the M mark, the equation
(i) should have the correct number of terms
(ii) be dimensionally correct i.e. all the terms need to be dimensionally correct
e.g. in a moments equation, every term must be a ‘force x distance’ term or ‘mass x distance’, if we allow them to cancel ‘g’ s.
For a resolution, all terms that need to be resolved (multiplied by sin or cos) must be resolved to earn the M mark.

M marks are sometimes dependent (DM) on previous M marks having been earned.
e.g. when two simultaneous equations have been set up by, for example, resolving in two directions and there is then an M mark for solving the equations to find a particular quantity – this M mark is often dependent on the two previous M marks having been earned.

‘A’ marks
These are dependent accuracy (or sometimes answer) marks and can only be awarded if the previous M mark has been earned. e.g. M0 A1 is impossible.

‘B’ marks
These are independent accuracy marks where there is no method (e.g. often given for a comment or for a graph).

A few of the A and B marks may be f.t. – follow through – marks.

3. General Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC – special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp – decimal places
- sf – significant figures
- ∗ – The answer is printed on the paper
- ⎿ – The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao), unless shown, for example as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
   - If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
   - If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer.

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general priniciples)

Method mark for solving 3 term quadratic:

1. **Factorisation**
   $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$ leading to $x = \ldots$
   $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$ leading to $x = \ldots$

2. **Formula**
   Attempt to use the correct formula (with values for *a*, *b* and *c*).

3. **Completing the square**
   Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0, q \neq 0$, leading to $x = \ldots$

Method mark for differentiation and integration:

1. **Differentiation**
   Power of at least one term decreased by 1. ($x^n \to x^{n-1}$)

2. **Integration**
   Power of at least one term increased by 1. ($x^n \to x^{n+1}$)

Use of a formula
Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers
Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Question Number	Scheme	Notes	Marks
1(a)	$\frac{dy}{dx} = 3\arcsin 2x + 3x\frac{1}{\sqrt{1-(2x)^2}} \times 2$ $\left(= 3\arcsin 2x + \frac{6x}{\sqrt{1-4x^2}}\right)$	M1: Obtains $p\arcsin qx + \frac{rx}{\sqrt{1-(sx)^2}}$ **or** $p\arcsin qx + \frac{rx}{\sqrt{1-tx^2}}$ $p, q, r, s, t > 0$ A1: Correct derivative. Allow unsimplified and isw. Allow $\sin^{-1}$ and condone "arsin" but "arsinh" or "arcsinh" is M0	M1A1
(b)	$x = \frac{1}{4} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2} + \sqrt{3}$	$\frac{\pi}{2} + \sqrt{3}$ only but allow $\frac{1}{2}\pi$ or 0.5π. Terms as a sum in either order. Allow $a = \frac{1}{2}$, $b = \sqrt{3}$ Isw following a correct answer.	B1**dep**
	This is a "Hence" question so this mark can only be awarded following full marks in part (a)		
			Total 3

Question Number	Scheme	Notes	Marks
2(a)	$x=-\frac{4}{3}$	$x=-\frac{4}{3}$ or any equivalent **equation**. Allow $x=\pm\frac{4}{3}$	B1
			(1)
(b)(i) **Way 1**	$\frac{a}{e}=\frac{4}{3}$ $b^2=a^2\left(e^2-1\right)\Rightarrow 5=a^2\left(\frac{9a^2}{16}-1\right)$	Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in *a*. Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$9a^4-16a^2-80=0$ $\Rightarrow\left(9a^2+20\right)\left(a^2-4\right)=0\Rightarrow a^2=...$	Solves a 3TQ in a^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of a^2 or *a* correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., "$\left(9a^2+20\right)\left(a^2-4\right)=0\Rightarrow a=4$" **Requires previous M mark.**	**d**M1
	$a=2$	Not $a=\pm2$ unless negative rejected	A1
			(3)
Way 2	$\frac{a}{e}=\frac{4}{3}$ $b^2=a^2\left(e^2-1\right)\Rightarrow 5=\left(\frac{4e}{3}\right)^2\left(e^2-1\right)$	Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in *e*. Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$16e^4-16e^2-45=0$ $\Rightarrow\left(4e^2-9\right)\left(4e^2+5\right)=0\Rightarrow e^2=...$	Solves a 3TQ in e^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of e^2 or *e* correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., "$\left(4e^2-9\right)\left(4e^2+5\right)=0\Rightarrow e=\frac{9}{4}$" **Requires previous M mark.**	**d**M1
	$\left(e=\frac{3}{2}\Rightarrow\right)\quad a=2$	Not $a=\pm2$ unless negative rejected but condone sight of "$e=\pm\frac{3}{2}$" or "$e=-\frac{3}{2}$"	A1
			(3)

Question Number	Scheme	Notes	Marks
2(b)(ii)	$e=\frac{3}{2} \Rightarrow ae=\frac{3}{2}\times 2$ or $ae=\frac{3a^2}{4}=\frac{3}{4}\times 4$ or $ae=c=\sqrt{a^2+b^2}=\sqrt{2^2+5}$	Uses a correct method to obtain a numerical expression for *ae* oe with their values of *a*, *e*, a^2, b^2 etc. however obtained. Condone use of a negative *e* or *a*	M1
	Foci are $(\pm 3, 0)$	Both correct foci as coordinates	A1
	Allow "$\frac{a}{e}=\frac{4}{3} \Rightarrow a=4,\ e=3$" to access the last M mark only in (b) for $(\pm 12, 0)$ provided the values of both *a* and *e* are clearly seen beforehand		**(2)**
			Total 6
	Note that it is possible to answer (ii) before (i) – e.g., Let foci be $(\pm c,\ 0)$ $a^2e^2=c^2=b^2+a^2=5+a^2$ and $\frac{a}{e}=\frac{a^2}{ae}=\frac{a^2}{c}=\frac{4}{3} \Rightarrow a^2=\frac{4}{3}c$ $\Rightarrow c^2=5+\frac{4}{3}c$ ((i) M1: Uses correct formulae to form an equation in *c* – condone b^2 replaced with 25 as with main scheme) $\Rightarrow 3c^2-4c-15=0 \Rightarrow (3c+5)(c-3)=0 \Rightarrow c=3$ ((i) **d**M1: Solves 3TQ to find positive real root) $\Rightarrow (\pm 3,\ 0)$ ((i)A1: Correct foci as coordinates) $a=\sqrt{\frac{4}{3}\times 3}$ ((ii) M1: Correct method for *a*) *a* = 2 ((ii) A1: Correct value)		

Question Number	Scheme	Notes	Marks
3 **Way 1** **Converts to sinh and cosh**	$4\tanh x-\text{sech}\, x=1$ $4\frac{\sinh x}{\cosh x}-\frac{1}{\cosh x}=1$ $4\sinh x-1-\cosh x=0$ $4\frac{e^{x}-e^{-x}}{2}-1-\frac{e^{x}+e^{-x}}{2}=0$	Replaces **one** hyperbolic function with its correct exponential equivalent. Allow for correct replacement of just e.g., $\sinh x$ after using $\tanh x=\frac{\sinh x}{\cosh x}$ May follow errors but do not allow any further marks if the original equation was reduced to one in a single hyperbolic function.	M1
	$3e^{2x}-2e^{x}-5=0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^{x}=\frac{2\pm\sqrt{4+60}}{6}\ \left(\Rightarrow\frac{2+8}{6}=\frac{5}{3}\right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3},\ \ln1\frac{2}{3},\ \ln1.\dot{6}$ only but allow k = ... No unrejected extra solutions	A1
			Total 6
Way 2 **Straight to e^x**	$4\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}-\frac{2}{e^{x}+e^{-x}}=1$	Replaces **one** hyperbolic function with its correct exponential equivalent	M1
	$3e^{2x}-2e^{x}-5=0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^{x}=\frac{2\pm\sqrt{4+60}}{6}\ \left(\Rightarrow\frac{2+8}{6}=\frac{5}{3}\right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3},\ \ln1\frac{2}{3},\ \ln1.\dot{6}$ only but allow k = ... No unrejected extra solutions	A1
			Total 6
	In Ways 1 & 2, if they form an equation which is not a quadratic in e^x they must achieve the correct exact root of $\frac{5}{3}$ to access the middle four marks		

Question Number	Scheme	Notes	Marks
3 **Way 3a** **Squaring (sinh)**	$4\sinh x-1=\cosh x$ $16\sinh^2 x-8\sinh x+1=\cosh^2 x$ $16\sinh^2 x-8\sinh x+1=1+\sinh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sinh x	M1
	$15\sinh^2 x-8\sinh x=0$	M1: Obtains a 2TQ with no constant or 3TQ in sinh x A1: Correct 2TQ	M1 A1
	$\sinh x=\frac{8}{15}$	Solves 2TQ (with no constant) or 3TQ in sinh x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x=\text{arsinh}\frac{8}{15}=\ln\left(\frac{8}{15}+\sqrt{\left(\frac{8}{15}\right)^2+1}\right)$ or $15e^{2x}-16e^x-15=0\Rightarrow$ $e^x=\frac{16\pm\sqrt{256+900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3},\ \ln 1\frac{2}{3},\ \ln 1.\dot{6}$ only but allow k = ... No unrejected extra solutions	A1
			Total 6
Way 3b **Squaring (sech)**	$4\tanh x=1+\text{sech}\, x$ $16\tanh^2 x=1+2\,\text{sech}\, x+\text{sech}^2 x$ $16\left(1-\text{sech}^2 x\right)=1+2\,\text{sech}\, x+\text{sech}^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sech x	M1
	$17\,\text{sech}^2 x+2\,\text{sech}\, x-15=0$	M1: Obtains a 2TQ (with no constant) or 3TQ in sech x A1: Correct 3TQ	M1 A1
	$\left(17\,\text{sech}\, x-15\right)(\text{sech}\, x+1)=0$ $\text{sech}\, x=\frac{15}{17}$	Solves 2TQ with no constant or 3TQ in sech x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x=\text{arcosh}\frac{17}{15}=\ln\left(\frac{17}{15}+\sqrt{\left(\frac{17}{15}\right)^2-1}\right)$ or $15e^{2x}-34e^x+15=0\Rightarrow$ $e^x=\frac{34\pm\sqrt{1156-900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3},\ \ln 1\frac{2}{3},\ \ln 1.\dot{6}$ only but allow k = ... No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
3 **Way 3c** **Squaring (tanh)**	$4\tanh x - 1 = \text{sech}\, x$ $16\tanh^2 x - 8\tanh x + 1 = \text{sech}^2 x$ $16\tanh^2 x - 8\tanh x + 1 = 1 - \tanh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in tanh *x*	M1
	$17\tanh^2 x - 8\tanh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in tanh *x* A1: Correct 2TQ	M1 A1
	$\tanh x = \frac{8}{17}$	Solves 2TQ with no constant or 3TQ in tanh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{artanh}\frac{8}{17} = \frac{1}{2}\ln\left(\frac{1+\frac{8}{17}}{1-\frac{8}{17}}\right)$ or $9e^{2x} - 25 = 0 \Rightarrow$ $e^x = \frac{5}{3}$	A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... No unrejected extra solutions	A1
			Total 6
Way 3d **Squaring (cosh)**	$4\sinh x = 1 + \cosh x$ $16\sinh^2 x = 1 + 2\cosh x + \cosh^2 x$ $16\cosh^2 x - 16 = 1 + 2\cosh x + \cosh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in cosh *x*	M1
	$15\cosh^2 x - 2\cosh x - 17 = 0$	M1: Obtains a 2TQ with no constant or 3TQ in cosh *x* A1: Correct 3TQ	M1 A1
	$(15\cosh x - 17)(\cosh x + 1) = 0$ $\cosh x = \frac{17}{15}$	Solves 2TQ (with no constant) or 3TQ in cosh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arcosh}\frac{17}{15} = \ln\left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1}\right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
4(a)	$\int \frac{1}{\sqrt{9x^2+16}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{16}{9}}}\,dx$ $= \frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)$ or $\frac{1}{3}\text{arsinh}\left(\frac{x}{\frac{4}{3}}\right)$ $(+c)$ or $\frac{1}{3}\ln\left(x+\sqrt{x^2+\left(\frac{4}{3}\right)^2}\right)$ $(+c)$	M1: Obtains $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ A1: Any correct expression. Could be unsimplified and isw. The "+*c*" is not required. Allow $\sinh^{-1}$ and condone "arcsinh". "arcsin" or "arsin" is M0	M1 A1
			(2)
(b)	$\int_{-2}^{2} \frac{1}{\sqrt{9x^2+16}}\,dx$ $= \left[\frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{-2}^{2}$ or $\left[\frac{2}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{0}^{2}$ $= \frac{1}{3}\text{arsinh}\left(\frac{3\times 2}{4}\right)-\frac{1}{3}\text{arsinh}\left(\frac{3\times -2}{4}\right)$ or $\frac{2}{3}\text{arsinh}\left(\frac{3}{2}\right)$ OR $\left[\frac{1}{3}\ln\left(x+\sqrt{x^2+\frac{16}{9}}\right)\right]_{-2}^{2}$ $= \frac{1}{3}\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right)-\frac{1}{3}\ln\left(-2+\sqrt{(-2)^2+\frac{16}{9}}\right)$ or $\frac{2}{3}\left(\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right)-\ln\left(0+\sqrt{0^2+\frac{16}{9}}\right)\right)$	Substitutes the limits 2 and -2 into an expression of the form $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ and subtracts either way round or obtains an expression for $2[...]_{0}^{\pm 2}$ The expression does not have to be consistent with their answer to (a). No rounded decimals unless exact values recovered. Any f(0) = 0 can be implied by omission. Condone poor bracketing.	M1
	$\frac{1}{3}\ln\left(\frac{11}{2}+\frac{3\sqrt{13}}{2}\right)$ or $\frac{1}{3}\ln\frac{11+3\sqrt{13}}{2}$ or $\frac{2}{3}\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)$ or $\frac{2}{3}\ln\frac{3+\sqrt{13}}{2}$	**d**M1: Obtains an expression of the form $a\ln\left(b+c\sqrt{13}\right)$ *or* $a\ln\left(\frac{d+e\sqrt{13}}{f}\right)$ where a,b,c,d,e,f are exact and >0. Condone poor bracketing. **Requires previous M mark.** A1: Any correct equivalent in an appropriate form (fractions may not be in simplest form) with correct bracketing if necessary and isw. **Must come from correct work**. Allow e.g., $a=\frac{2}{3}$, $b=\frac{3}{2}$, $c=\frac{1}{2}$	**d**M1 A1
	For information the decimal answer is 0.7965038115		**(3)**
			Total 5

Question Number	Scheme	Notes	Marks
5(a)(i) **Way 1** $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert = \begin{vmatrix} a-\lambda & a & 1 \\ -a & 4-\lambda & 0 \\ 4 & a & 5-\lambda \end{vmatrix}$ $=(a-\lambda)(4-\lambda)(5-\lambda)-a\times -a(5-\lambda)+\left(-a^2-4(4-\lambda)\right)$ or $\lvert\mathbf{A}-2\mathbf{I}\rvert = \begin{vmatrix} a-2 & a & 1 \\ -a & 2 & 0 \\ 4 & a & 3 \end{vmatrix}$ $=6(a-2)-a\times -3a+\left(-a^2-8\right)$	Obtains an expression for $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$ in terms of a and λ or just a if λ is replaced by 2. If method unclear insist on 2 out of 3 correct parts. May multiply along any row/column. Sarrus leads to the same expressions shown (or the expressions all multiplied by -1).	M1
	$\lambda=2 \Rightarrow (a-2)\times 2\times 3+3a^2-a^2-8=0$ $2a^2+6a-20=0 \Rightarrow a^2+3a-10=0$ $\Rightarrow (a-2)(a+5)=0 \Rightarrow a=...$	Following use of $\lambda=2$, forms and solves a 3TQ in *a*. Apply usual rules. If no working they must obtain one correct solution for their 3TQ which could be complex. Could be implied. **Requires previous M mark.**	**d**M1
	$(a>0 \Rightarrow) a=2$	Correct value of *a* from correct work. If –5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If *a* = 2 is arrived at fortuitously, all marks are available for the remainder of the question		**(3)**
Way 2 $\mathbf{Ax}=2\mathbf{x}$	$\mathbf{Ax}=2\mathbf{x}\Rightarrow$ $ax+ay+z=2x$ $-ax+4y=2y$ $4x+ay+5z=2z$	Uses $\mathbf{Ax}=2\mathbf{x}\ \left[or\ (\mathbf{A}-2\mathbf{I})\mathbf{x}=0\right]$ to obtain three simultaneous equations. Allow if given as two equal vectors.	M1
	$\Rightarrow a^2+3a-10=0$ $\Rightarrow (a-2)(a+5)=0 \Rightarrow a=...$	Forms and solves a 3TQ in *a*. Apply usual rules. If calculator used must obtain one correct solution for their 3TQ which could be complex. Could be implied. **Requires previous M mark.**	**d**M1
	$(a>0 \Rightarrow) a=2$	Correct value of *a* from correct work. If –5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If *a* = 2 is arrived at fortuitously, all marks are available for the remainder of the question		**(3)**
(a)(ii)	$(2-\lambda)(4-\lambda)(5-\lambda)+4(5-\lambda)+(-4-16+4\lambda)=0$ $\Rightarrow (5-\lambda)\left[(2-\lambda)(4-\lambda)+4-4\right]=0$ $\Rightarrow (5-\lambda)(2-\lambda)(4-\lambda)=0 \Rightarrow \lambda=...$	Uses their value of *a* in a recognisable attempt at a characteristic equation and achieves a real non-zero eigenvalue $\neq 2$. There must be some algebra but it may be poor.	M1
	4 and 5	Both correct (no extra) and from correct work	A1
	For information the cubic is $\pm\left(\lambda^3-11\lambda^2+38\lambda-40\right)=0$		**(2)**

Question Number	Scheme	Notes	Marks
5(b)	$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}="4"\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-"4"\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$	$2x+2y+z=4x$, $-2x+4y=4y$, $4x+2y+5z=4z$ *or* $-2x+2y+z=0$, $-2x=0$, $4x+2y+z=0$	
	OR		
	$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}="5"\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-"5"\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$	$2x+2y+z=5x$, $-2x+4y=5y$, $4x+2y+5z=5z$ *or* $-3x+2y+z=0$, $-2x-y=0$, $4x+2y+z=0$	M1
	Uses $\mathbf{Ax}=\lambda\mathbf{x}$ *or* $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$ with their value of *a* and a real non-zero value of $\lambda\neq 2$ to obtain three simultaneous equations (allow if given as two equal vectors) Alternatively attempts vector product of two rows of $\mathbf{A}-"4"\mathbf{I}$		
	$\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **or** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$	One correct eigenvector. As shown or multiple or with components multiplied by e.g. "*k*" Accept e.g., $x=0,\ y=-1,\ z=2$	A1
	$\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **and** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$	Both correct eigenvectors. As shown or multiple or with components multiplied by e.g. *k* Accept $x=...,\ y=...,\ z=...$ Both these 2 A marks could be implied by their normalised eigenvectors	A1
	$\left[\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\text{was given}\right]$ $\pm\frac{1}{\sqrt{5}}\begin{pmatrix}0\\-1\\2\end{pmatrix},\ \pm\frac{1}{\sqrt{54}}\begin{pmatrix}1\\-2\\7\end{pmatrix}$ oe	M1: A correct method to normalise at least one of their eigenvectors A1: Both correct. Allow any exact equivalents. Isw	M1 A1
	All marks available regardless of how $a=2,\ \lambda_2=4$ & $\lambda_3=5$ have been obtained		**(5)**
(c)	$\mathbf{P}=$ e.g. $\begin{pmatrix}\frac{1}{\sqrt{6}} & "0" & "\frac{1}{\sqrt{54}}"\\ \frac{1}{\sqrt{6}} & "\frac{-1}{\sqrt{5}}" & "\frac{-2}{\sqrt{54}}"\\ \frac{-2}{\sqrt{6}} & "\frac{2}{\sqrt{5}}" & "\frac{7}{\sqrt{54}}"\end{pmatrix}$ or $\begin{pmatrix}\frac{\sqrt{6}}{6} & "0" & "\frac{\sqrt{6}}{18}"\\ \frac{\sqrt{6}}{6} & "\frac{-\sqrt{5}}{5}" & "\frac{-\sqrt{6}}{9}"\\ \frac{-\sqrt{6}}{3} & "\frac{2\sqrt{5}}{5}" & "\frac{7\sqrt{6}}{18}"\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & "4" & 0\\ 0 & 0 & "5"\end{pmatrix}$	1st B1ft: A correct ft **P** or **D**. If awarding for **P** there must have been an attempt to normalise both their eigenvectors – accept if the components were divided by $k\neq 1$. Labels may be absent /wrong. 2nd B1ft: Correct ft **P** and **D**. They must be consistent and labelled correctly. Isw once correct ft matrices are seen. Note that any of the columns of **P** could be in the opposite direction. **Award B1 B0 for one correct ft matrix** **B1 B1 for both ft matrices**	B1ft B1ft
	Note: Due to the issue with this question part, the final mark can be awarded for an attempt at **D** = $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ by multiplication provided <u>one correct unsimplified element in the correct position on the leading diagonal is obtained</u>. For example: $\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{54}} & -\frac{2}{\sqrt{54}} & \frac{7}{\sqrt{54}}\end{pmatrix}\begin{pmatrix}2 & 2 & 1\\ -2 & 4 & 0\\ 4 & 2 & 5\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{54}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{54}}\\ -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{7}{\sqrt{54}}\end{pmatrix}=\begin{pmatrix}2 & -\frac{2\sqrt{30}}{3} & -\frac{25}{6}\\ -\frac{\sqrt{30}}{3} & 4 & \frac{8\sqrt{30}}{9}\\ -\frac{5}{3} & \frac{32\sqrt{30}}{45} & 5\end{pmatrix}$		
			(2)
			Total 12

Question Number	Scheme	Notes	Marks
6(a)	$\frac{dx}{d\theta}=\begin{cases}a(1-\cos\theta)\\ \text{or}\\ a-a\cos\theta\end{cases}$ **or** $\frac{dy}{d\theta}=a\sin\theta$	At least **one** correct derivative	B1
	$a^2(1-\cos\theta)^2+(a\sin\theta)^2$ $=a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$ $=2a^2(1-\cos\theta)$	Squares and adds their derivatives and uses $\cos^2\theta+\sin^2\theta=1$ to obtain an expression in $\cos\theta$ only (not $\cos^2\theta$) Could be implied	M1
	$=2a^2\left(1-\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)\right)=4a^2\sin^2\frac{\theta}{2}$	**d**M1: Replaces $\cos\theta$ with $\pm1\ \pm2\sin^2\frac{\theta}{2}$ or equivalent trig work (sign errors only on identities) to obtain an expression in $\sin^2\frac{\theta}{2}$ only **Requires previous M mark.** Can be implied. A1: Achieves $4a^2\sin^2\frac{\theta}{2}$ or k = 4 from correct work	**d**M1 A1
			(4)
(b)	$\text{S.A.}=(2\pi)\int y\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}d\theta$ $=(2\pi)\int_{(0)}^{(2\pi)}a(1-\cos\theta)\left(2a\sin\frac{\theta}{2}\right)d\theta$	Applies $y\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}$ with their $ka^2\sin^2\frac{\theta}{2}$ and square roots. The result of the square root may be incorrect but must be of the form $p\sin\frac{\theta}{2}$ Allow a slip replacing y but they must not have used x, $\frac{dx}{d\theta}$ or $\frac{dy}{d\theta}$ for y Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required.	M1
	$=(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos\theta\right)d\theta$ $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)\right)d\theta$ or e.g., $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}2\sin^3\frac{\theta}{2}\,d\theta$	Uses trig identity/identities (condoning sign errors) to obtain an expression with arguments of $\frac{\theta}{2}$ only. Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required. **Dependent on previous M mark.**	**d**M1
	Scheme continues overleaf...		

Question Number	Scheme	Notes	Marks
6(b) **cont.**	$\left(= (2\pi)4a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta\right)$ $S = 8\pi a^2\left[-2\cos\frac{\theta}{2} + \frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ or e.g., $\pi a^2\left[-16\cos\frac{\theta}{2} + \frac{16}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$	A correct expression for the surface area ignoring limits ft their numerical *k*, i.e., $S = 2k\pi a^2\left[-2\cos\frac{\theta}{2} + \frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ oe If they integrate in a piecemeal fashion, award this mark if they have a correct expression for their *k* when integration is completed – any partial evaluations must be correct for their *k*	A1ft
	$= 8\pi a^2\left[\left(-2\cos\frac{2\pi}{2} + \frac{2}{3}\cos^3\frac{2\pi}{2}\right) - \left(-2\cos 0 + \frac{2}{3}\cos^3 0\right)\right]$	Substitutes correct limits and attempts to subtract either way round following a completed attempt at integration with a numerical *k*. **Requires previous M marks** and must have used 2π . Look for evidence of correct limit substitution and subtraction. There may be slips but insist on limits being applied on all integrations if they have been carried out separately. Algebraic results of integration must be seen	**dd**M1
	$= \frac{64}{3}\pi a^2$	Correct exact answer. Accept equivalent fractions.	A1
	All marks available regardless of how $k = 4$ was obtained		**(5)**
			Total 9
	Other integration methods: Allow the second M mark to be available before any attempt at integration is made. Otherwise the second M is only awarded if they **complete** integration without any loss of the required forms (i.e., sign and coefficient errors only and just sign errors only with any trig identities). The first A (ft) mark is for a fully correct expression ignoring limits for their *k*. The last two marks are the same as the main scheme. For information: Applying parts to $\int \sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta$ gives $\frac{2}{3}\left(\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\right)$ Using addition formulae: $\int \sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta = \frac{1}{2}\int\left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right)\mathrm{d}\theta = \frac{1}{2}\left(2\cos\frac{\theta}{2} - \frac{2}{3}\cos\frac{3\theta}{2}\right)$		

Question Number	Scheme	Notes	Marks
7(a)	$\begin{pmatrix}0\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}=8\mathbf{i}-2\mathbf{j}-3\mathbf{k}$	M1: Attempts vector product of two vectors in the plane. Unless there is a full clear method they must achieve two correct components A1: $\pm(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ or multiple	M1 A1
	Allow any vector notation throughout this question		**(2)**
(b)	l has direction vector $\pm(2\mathbf{j}+\ 2\mathbf{k})$	Correct direction for l	B1
	$(\cos\alpha\ or\ \sin\theta=)$ $\left\|\frac{"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"."(2\mathbf{j}+2\mathbf{k})"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|=\left\|\frac{"(8)(0)+(-2)(2)+(-3)(2)"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{0^2+2^2+2^2}"}\right\|\left(=\left\|\frac{-10}{\sqrt{77}\times\sqrt{8}}\right\|\ or\ \left\|\frac{-5\sqrt{154}}{154}\right\|\right)$ M1: For the scalar product of their normal and direction vector divided by the product of the magnitudes of their vectors. The first expression above oe is sufficient. There must have been a valid attempt at both vectors. Allow copying errors/slips if intention is clear. Modulus not required. A1ft: A correct ft numerical expression with scalar product calculated as shown by second expression or better. Allow a decimal correct to 2sf. Modulus not required. Ignore labelling. Actual decimal is 0.40291148... Implied by awrt 24 or 66 or 114 provided some work and nothing incorrect seen. Allow awrt 0.41, 1.16 or 1.99 if working in radians.		M1 A1ft
	Acute angle between l and P $=90-\alpha=90-66.23968409...$ or $\theta=23.76031591...\Rightarrow 24°$ to the nearest degree	awrt 24 from correct work which could be minimal. Degrees symbol not required. **Mark final answer.**	A1
			(4)
	Note that a vector product could be used: M1: $\left\|\frac{\left\|"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"\times"(2\mathbf{j}+2\mathbf{k})"\right\|}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|$ A1: $\left\|\frac{"\sqrt{2^2+16^2+16^2}"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|\ \left(=\frac{2\sqrt{129}}{\sqrt{77}\sqrt{8}}=0.9152389511...\right)$ The modulus of the numerator is required for any marks		
(c) **Way 1** **Parallel planes**	$(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=-5$ or $(6\mathbf{i}-3\mathbf{j}-6\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=72$	M1: Finds a value for the scalar product of a position vector of a point in the plane or the given point and their normal. A1: –5 or 72 (or 5 or -72 if normal is in the opposite direction). May be seen as a distance e.g., $\frac{-5}{\sqrt{"77"}}$	M1 A1
	Shortest distance is $\left\|\frac{-5-72}{\sqrt{77}}\right\|=\frac{77}{\sqrt{77}}$ or $\sqrt{77}$	**d**M1: Having attempted both scalar products, obtains a numerical expression for the distance. Award for $\frac{\pm"5"\pm"72"}{\sqrt{"8"^2+"2"^2+"3"^2}}$ **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)

Question Number	Scheme	Notes	Marks
7(c) **Way 2** **Perp. distance formula**	$(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(\text{"}8\mathbf{i}-2\mathbf{j}-3\mathbf{k}\text{"})=-5$	M1: Finds a value for the scalar product of a position vector to a point the plane and their normal. A1: -5 (or 5 if normal is in the opposite direction)	M1 A1
	"$8x-2y-3z+5=0$" Shortest distance is $\frac{\lvert(\text{"}8\text{"})(6)+(\text{"}-2\text{"})(-3)+(\text{"}-3\text{"})(-6)+\text{"}5\text{"}\rvert}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}$ $=\frac{77}{\sqrt{77}}$ or $\sqrt{77}$	**d**M1: Uses distance formula with their normal and plane equation to reach a numerical expression for the distance. Condone sign slip on their -5 and their *d* must not be zero. **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
Way 3 **Projection /resolving formula**	Let Q be the point on the plane (1, 2, 3) then $\overrightarrow{PQ}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})-(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})$ $=-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}$	M1: Attempts vector from given point to a point on the plane A1: Correct vector $(\pm)$	M1 A1
	Shortest distance is $\lvert\overrightarrow{PQ}.\mathbf{n}\rvert=$ $\frac{\lvert(\text{"}-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}\text{"}).(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\rvert}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}=\ldots$ $=\frac{77}{\sqrt{77}}$ *or* $\sqrt{77}$	**d**M1: Uses formula with their vectors to reach a numerical expression for the distance **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
Way 4 **Example of method involving the point where the line meets plane**	Line through given point in direction of normal is $r=(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})+\lambda(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ & meets plane "$8x-2y-3z+5=0$" when $8(6+8\lambda)-2(-3-2\lambda)-3(-6-3\lambda)+5=0$ $\Rightarrow\lambda=-1$	M1: Uses line through given point in the direction of their normal and substitutes into their plane to find a value for λ. The *d* in their plane equation must not be zero A1: Correct value	M1 A1
	$\lvert-1(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\rvert=\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}$ Or point of intersection is (6 –"8", –3 –"–2", –6 –"–3") = (–2, –1, –3) and distance is $\sqrt{(6-\text{"}-2\text{"})^2+(-3-\text{"}-1\text{"})^2+(-6-\text{"}-3\text{"})^2}$ $\Rightarrow\sqrt{77}$	**d**M1: Attempts $\lvert\lambda\mathbf{n}\rvert$ or finds point on the plane and obtains numerical expression for distance between this point and the given point **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
	Marks are scored through the Way which is the best overall match for the attempt. Credit for work done in (b) is only available for part (c) if it is used in part (c).		
			Total 10

Question Number	Scheme	Notes	Marks
8(a) **Way 1**	$I_n = \int \cos^n x\,dx = \int \cos x \cos^{n-1} x\,(dx)$	Correct split. Could be implied by their work	M1
	$= \sin x\cos^{n-1} x + \int (n-1)\cos^{n-2} x \sin^2 x\,(dx)$	Obtains $p\sin x\cos^{n-1} x + \int q\cos^{n-2} x\sin^2 x\,(dx)$ oe **Requires previous M mark.**	**d**M1
	$= \sin x\cos^{n-1} x + \int (n-1)\cos^{n-2} x\left(1-\cos^2 x\right)(dx)$	Replaces $\sin^2 x$ with $1-\cos^2 x$ to achieve a correct expression for I_n	A1
	$= \sin x\cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$ $\Rightarrow I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}$ *	Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "dx"s but there must be no missing arguments. Any clear bracketing error must be recovered **before** given answer.	A1*
			(4)
Way 2	$I_n = \int \cos^n x\,dx = \int \cos^2 x\cos^{n-2} x\,(dx)$ $= \int \left(1-\sin^2 x\right)\cos^{n-2} x\,(dx)$	Correct split **and** replaces $\cos^2 x$ with $1-\sin^2 x$	M1
	$= \int \left(\cos^{n-2} x - \cos^{n-2} x\sin^2 x\right)(dx)$ $= \int \cos^{n-2} x\,(dx) - \int \left(\sin x\sin x\cos^{n-2} x\right)(dx) = ...$ M1: Expands, splits and obtains $p\int \cos^{n-2} x\,(dx) + q\cos^{n-1} x\sin x + \int r\cos^n x\,(dx)$ oe **Requires previous M mark.** A1: Correct expression for I_n: $\int \cos^{n-2} x\,(dx) - \left(-\frac{1}{n-1}\cos^{n-1} x\sin x + \int \frac{1}{n-1}\cos^n x\,(dx)\right)$ oe		**d**M1 A1
	$= I_{n-2} + \frac{1}{n-1}\cos^{n-1} x\sin x - \frac{1}{n-1}I_n$ $\Rightarrow I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}$ *	Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "dx"s but there must be no missing arguments. Any bracketing error must be recovered **before** given answer.	A1*
			(4)
(b)	$I_n = \frac{1}{n}\left[\cos^{n-1} x\sin x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}I_{n-2}$ or $= \frac{1}{n}(n-1)I_{n-2}$ $I_2 = \frac{1}{2}\left[\cos^{2-1} x\sin x\right]_0^{\frac{\pi}{2}} + \frac{2-1}{2}I_0$ or $= \frac{1}{2}I_0$	Uses the RF to obtain an expression for I_n in terms of I_{n-2} or I_2 in terms of I_0 Condone if necessary if limits are absent.	M1
	$I_n = \frac{(n-1)(n-3)...5\times3\times1}{n(n-2)(n-4)...6\times4\times2}I_0$ with dots & at least 3 terms in each product (first 2 & last, or first & last 2)	Correct expression for I_n in terms of I_0 oe following correct work including 2 applications of the reduction formula (which could be embedded) **prior** to this answer. I_0 may have been calculated previously but do not allow just the final printed answer to imply this mark.	A1
	e.g., $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$ or $I_0 = \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ or $I_0 = \frac{\pi}{2} - 0$	Correct value for I_0 - requires written evidence of integration (minimal)	B1
	$\therefore I_n = \frac{(n-1)(n-3)...5\times3\times1}{n(n-2)(n-4)...6\times4\times2}\times\frac{\pi}{2}$ * Allow extra terms in both products.	Proceeds to given answer. Requires all previous marks. Withhold this mark if no $\frac{1}{k}\left[\cos^{k-1} x\sin x\right]_0^{\frac{\pi}{2}}$ is seen or expression just disappears – one such expression must be replaced by "0" or have substitution seen	A1*

	Attempts via proof by induction will be reviewed.		**(4)**
	Attempts may be seen via $I_n = \frac{(n-1)(n-3)...3}{n(n-2)...4} I_2$ and $I_2 = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2} \times \frac{\pi}{2}$		
Question Number	Scheme	Notes	Marks
8(c)	$\int_0^{\frac{\pi}{2}} \cos^6 x \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^6 x \left(1 - \cos^2 x\right) dx$	Replaces $\sin^2 x$ with $1 - \cos^2 x$ Can be implied by an attempt at $I_6 - I_8$	M1
	$= I_6 - I_8 = \left(\frac{5 \times 3 \times 1}{6 \times 4 \times 2} - \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2}\right)\frac{\pi}{2}$	Any correct numerical expression for the integral	A1
	$\left(= \frac{5}{32}\pi - \frac{35}{256}\pi =\right)\frac{5}{256}\pi$ oe	Correct exact value. Accept equivalent fractions and allow e.g., $\left(\frac{5}{128}\right)\frac{\pi}{2}$	A1
	This is a "Hence" and requires clear use of $I_6 - I_8$ **For the A marks there must be no evidence that the answer has been arrived at without using part (b). There is no credit in (b) for work in (c).** **Just "$I = \frac{5}{256}\pi$" is 0/3 but just "$I_6 - I_8 = \frac{5}{256}\pi$" is 3/3**		
			(3)
			Total 11

Question Number	Scheme	Notes	Marks
9(a)(i)	$b^2=a^2\left(1-e^2\right)\Rightarrow 1=9\left(1-e^2\right)$ $\Rightarrow e^2=...\left(\frac{8}{9}\right),\ e=\frac{2\sqrt{2}}{3}$ or $\frac{\sqrt{8}}{3}$	M1: Uses a correct eccentricity formula with correct values for a and b and obtains a value for e^2 or e A1: Correct value for e (not $\pm$) Could be implied	M1 A1
	Foci are $\left(\pm 2\sqrt{2},0\right)$ or $\left(\pm\sqrt{8},0\right)$	B1: Both correct foci as coordinates Condone any use of a negative e **Note that this is not an ft mark.**	B1
			(3)
(a)(ii)	$x=\pm\frac{9\sqrt{2}}{4}$ or $\pm\frac{9\sqrt{8}}{8}$ or $\pm\frac{9}{\sqrt{8}}$ oe Both correct **equations**. Requires single fraction. **Allow ft**: $x=\pm\frac{\mathbf{3}}{\text{their } e}$ computed into a single fraction, condoning e < 0 Allow "x_1 = ..., x_2 = ..." Condone, e.g., "$x=\pm\frac{a}{e}=\frac{9\sqrt{2}}{4}$ or $-\frac{9\sqrt{2}}{4}$" but just "$\frac{a}{e}=\pm\frac{9\sqrt{2}}{4}$" is B0		B1ft
			(2)
(b)	$\lvert PF_1\rvert=e\lvert PM_1\rvert$ or $\lvert PF_2\rvert=e\lvert PM_2\rvert$ oe	States this definition of an ellipse.	M1
Way 1 $PF=ePM$	$\lvert PF_1\rvert+\lvert PF_2\rvert=\underline{e\left(\lvert PM_1\rvert+\lvert PM_2\rvert\right)}$ or $\underline{e\left(\lvert M_1M_2\rvert\right)}$ "$\frac{2\sqrt{2}}{3}$"$\times 2\times$"$\frac{9\sqrt{2}}{4}$" oe or $\lvert PF_1\rvert+\lvert PF_2\rvert=$ $=$"$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"}-x\right)+$"$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"}+x\right)$	Correct method for a numerical expression (or with cancelling "x"s) for $\lvert PF_1\rvert+\lvert PF_2\rvert$ with their e and directrix. **One of the underlined expressions must be seen for the first approach.** **Requires previous M mark.**	**d**M1
	$=6$ *	Fully correct proof. Modulus signs are not required.	A1*
Way 1 **Guidance**	If they work in a and e, $e\times 2\times\frac{a}{e}$ is only acceptable if $\underline{e\left(\lvert PM_1\rvert+\lvert PM_2\rvert\right)}$ or $\underline{e\left(\lvert M_1M_2\rvert\right)}$ is seen (as with using the values) and $e\left(\frac{a}{e}-x\right)+e\left(\frac{a}{e}+x\right)\ (\Rightarrow 2a)$ is acceptable but note in both these general cases the second M mark becomes available when a = 3 is substituted. **<u>The second M is not available for any work which relies on</u>** $\lvert PF_1\rvert=\lvert PF_2\rvert$ **<u>Their proof needs to be shown to be valid for any position of *P*</u>** So $\lvert PF_1\rvert+\lvert PF_2\rvert=\frac{2\sqrt{2}}{3}\times\frac{9\sqrt{2}}{4}+\frac{2\sqrt{2}}{3}\times\frac{9\sqrt{2}}{4}$ or using $e\times\frac{a}{e}+e\times\frac{a}{e}$ cannot score the second M without $\underline{e\left(\lvert PM_1\rvert+\lvert PM_2\rvert\right)}$ or $\underline{e\left(\lvert M_1M_2\rvert\right)}$ being seen. If e appears as a value it must be correct for the final mark. Just $\lvert PF_1\rvert+\lvert PF_2\rvert=2a=2\times 3=6$ is 0/3		**(3)**

	Having earned the first mark in Way 1, some candidates proceed to work with a specific point on the ellipse as in Way 2. Further credit is only available if they clearly state e.g, " $\lvert PF_1\rvert + \lvert PF_2\rvert$ is constant for any *P*"		
Question Number	Scheme	Notes	Marks
9(b) **Way 2** $PF_1 + PF_2 = k$	$\lvert PF_1\rvert + \lvert PF_2\rvert = \lvert QF_1\rvert + \lvert QF_2\rvert$ where *P* and *Q* are any points on the ellipse oe	States this oe definition of an ellipse, justified by explanation. Accept e.g., "$\lvert PF_1\rvert + \lvert PF_2\rvert$ is constant for any *P*"	M1
	e.g. *Q* is where *E* crosses positive *x*-axis $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert = 3 - "2\sqrt{2}" + 3 + "2\sqrt{2}"$ *Q* is where *E* crosses positive *y*-axis $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert = 2\sqrt{1^2 + "2\sqrt{2}"^2}$ *Q* is on *E* directly above F_1 $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert =$ $\sqrt{1 - \frac{("2\sqrt{2}"^2)}{9}} + \sqrt{(2 \times "2\sqrt{2}")^2 + 1 - \frac{("2\sqrt{2}"^2)}{9}}$	Correct method for a numerical value for $\lvert PF_1\rvert + \lvert PF_2\rvert$ using another point on the ellipse and their foci. **Requires previous M mark.**	**d**M1
	$= 6$ *	Fully correct proof. Modulus signs are not required.	A1*
			(3)
Way 3 **Point in terms of** θ	$P(3\cos\theta, \sin\theta)$ $\lvert PF_1\rvert^2 = (3\cos\theta - "2\sqrt{2}")^2 + \sin^2\theta$ or $\lvert PF_2\rvert^2 = (3\cos\theta + "2\sqrt{2}")^2 + \sin^2\theta$	Correct general point in parametric form and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of *a*, *b* and θ	M1
	$\lvert PF_1\rvert + \lvert PF_2\rvert =$ $\sqrt{8\cos^2\theta - 12\sqrt{2}\cos\theta + 9} + \sqrt{8\cos^2\theta + 12\sqrt{2}\cos\theta + 9}$	Correct method for $\lvert PF_1\rvert + \lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when *a* and *b* are substituted. **Requires previous M mark.**	**d**M1
	$\lvert PF_1\rvert + \lvert PF_2\rvert =$ $3 - 2\sqrt{2}\cos\theta + 3 + 2\sqrt{2}\cos\theta = 6$*	Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.	A1*
			(3)
Way 4 **Point in terms of** ***x***	$P\left(x, \sqrt{1 - \frac{x^2}{9}}\right)$ or $P\left(x, \sqrt{\frac{9 - x^2}{9}}\right)$ $\lvert PF_1\rvert^2 = ("2\sqrt{2}" - x)^2 + 1 - \frac{x^2}{9}$ or $\lvert PF_2\rvert^2 = (x + "2\sqrt{2}")^2 + 1 - \frac{x^2}{9}$	Correct general point in terms of *x* and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of *a*, *b* and *x*.	M1
	$\lvert PF_1\rvert + \lvert PF_2\rvert = \sqrt{\frac{8}{9}x^2 - 4\sqrt{2}x + 9} + \sqrt{\frac{8}{9}x^2 + 4\sqrt{2}x + 9}$	Correct method for $\lvert PF_1\rvert + \lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when *a* and *b* are substituted. **Requires previous M mark.**	**d**M1

	$\lvert PF_1\rvert+\lvert PF_2\rvert=3-\frac{2\sqrt{2}}{3}x+3+\frac{2\sqrt{2}}{3}x=6$*	Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.	A1*
			(3)
	Creditworthy alternative approaches will be reviewed		
Question Number	Scheme	Notes	Marks
9(c)	$x^2+9(2x+c)^2=9$ or $\frac{x^2}{9}+(2x+c)^2=1$	Substitutes line into the ellipse equation. Condone slips provided intention clear.	M1
	$37x^2+36cx+9c^2-9=0$ or e.g., $\frac{37}{9}x^2+4cx+c^2-1=0$	Correct quadratic with x^2 terms collected (could be implied)	A1
	½ (sum of roots) $\Rightarrow$ $(x=)\frac{-18c}{37}$ or $(x=)\frac{1}{2}\left(\frac{-36c+\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}+\frac{-36c-\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}\right)$ M1: Correct attempt at ½ (sum of roots), i.e., $-\frac{b}{2a}$ for their quadratic. Ignore how the expression is labelled. **Requires previous M mark.** A1: Any correct **equation** in x and c Allow this mark if e.g., x is seen as M_x		**d**M1 A1
	$\Rightarrow c=\text{"}-\frac{37}{18}\text{"}x\Rightarrow y=2x+\left(\text{"}-\frac{37}{18}\text{"}\right)x$ or $x=\text{"}-\frac{18}{37}\text{"}c\Rightarrow y=2\times\text{"}-\frac{18}{37}\text{"}c+c\Rightarrow\ldots\left(y=\frac{c}{37}\Rightarrow\frac{y}{x}=-\frac{1}{18}\right)$	Substitutes their $c = px$ into the line to obtain an equation in ***x* and *y*** only. Allow e.g., x_M and y_M and condone e.g., suffixes of P & Q This may also be achieved by e.g., finding y in terms of c and then eliminating c with their equation in x and c Must not be using "M_x" or "M_y" etc. but imply this mark from a locus equation in x and y or $x_{\ldots}$ and $y_{\ldots}$ with appropriate suffixes **Requires both previous M marks**	**dd**M1
	$\Rightarrow y_{\ldots}=-\frac{1}{18}x_{\ldots}$ oe $\therefore$ l passes through the origin oe *	Obtains correct equation for locus (accept equivalents) **and** makes conclusion e.g., "passes/goes through origin/O/(0,0)" but allow "shown"/"as required"/"QED" etc. **Requires all previous marks.**	A1*
			(6)
			Total 13
			PAPER TOTAL: 75

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